

## CHAPTER 11 CURVE OF GROWTH

### 11.1 *Introduction*

The *curve of growth* (*die Wachstumskurve*) is a graph showing how the equivalent width of an absorption line, or the radiance of an emission line, increases with the number of atoms producing the line.

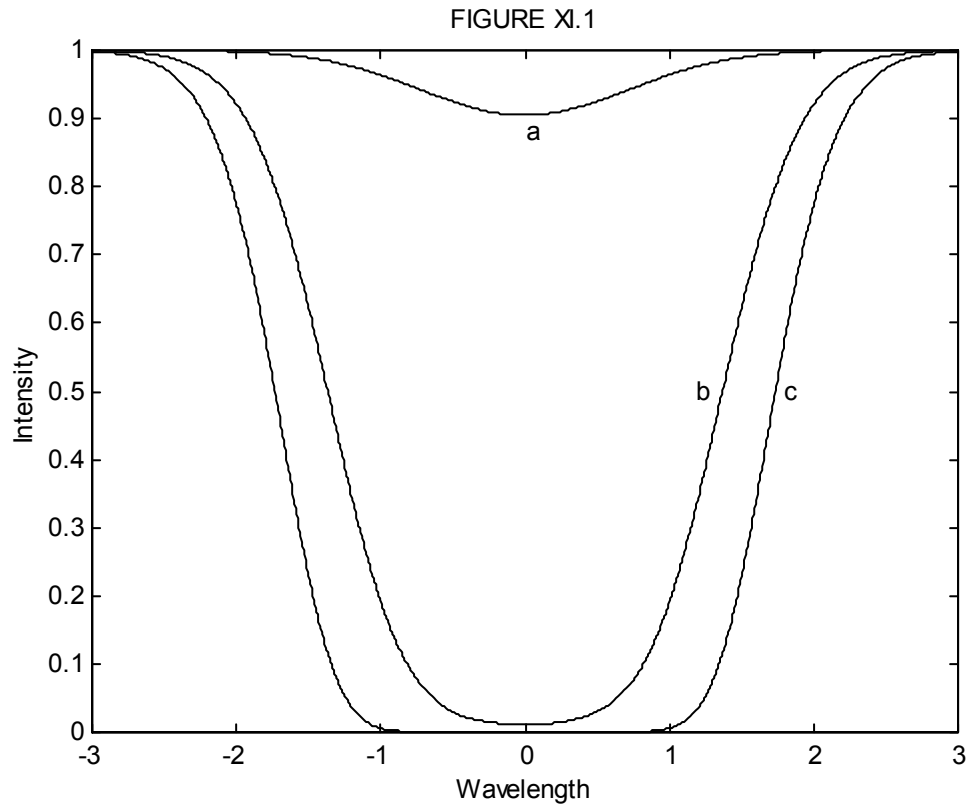
In an *optically thin* gas, the equivalent width of an absorption line, or the radiance of an emission line, is linearly proportional to the number of atoms in the initial level of the line. Let us imagine, for example, that we have a continuous source of radiation, and, in front of it, we have a homogeneous, isothermal slab of gas, and that there are  $N_1$  atoms per unit area in the line of sight in the lower level of some absorption line. We could double  $N_1$  either by doubling the thickness of the slab, or by doubling the density of the gas within the slab. Either way, if doubling the number of atoms per unit area (the column density) in level 1 results in a doubling of the equivalent width of the line, then the gas is said to be optically thin. More precisely, the *line* is optically thin – for there may well be other lines in the spectrum which are not optically thin. I suppose one could say that a gas is optically thin at the wavelength of a particular line if you can see all of the atoms – even those at the back. In Chapter 9, where we were developing formulas for the equivalent width of a line, and in Chapter 10, where we were studying line profiles, we were limiting our attention to optically thin lines. We shall depart from this assumption in this Chapter, although we shall still assume that our slab of gas is homogeneous (same temperature and pressure throughout) and in thermodynamic equilibrium.

We can see, by referring to figure XI.1, why it is that the equivalent width of an absorption line cannot continue to increase indefinitely and linearly as the column density increases. The figure represents the profile of an absorption line. (Strictly speaking, the ordinate should read “radiance per unit wavelength interval”, rather than “intensity”.) Profile *a* is a weak line (actually a gaussian profile) that is optically thin. In profile *b*, we have greatly increased the column density  $N_1$ , and we see that the intensity at the centre of the line is almost zero. Hardly any of the background light from the continuous source is getting through. At this wavelength, the background is black. Increasing the column density will make no difference at all to the central intensity, and hardly any change in the equivalent width. Thus a graph of equivalent width versus column density will no longer be a linearly increasing function, but will be almost horizontal.

But it will not remain horizontal. As we see in profile *c*, when yet further atoms have been added, although the central depth does not and cannot become any deeper, the wings of the profile start to add to the equivalent width, so that the equivalent width starts to increase again, although rather more slowly than during the optically thin stage. Thus we might expect three stages in the curve of growth. At first, the equivalent width increases linearly with the column density of absorbers. Then there will be a stage in which the

equivalent width is scarcely increasing. Finally, there will be a third stage in which the equivalent width increases, but not as rapidly as in the optically thin case.

It will also be noticed that, as soon as the profile (which was gaussian when it was optically thin) ceases to be optically thin, the profile becomes distorted and is no longer the same as it was in the optically thin region.



From our qualitative description of the curve of growth, it will be evident that the form of the curve of growth will depend on the form of the original line profile. For example, it will be recalled that the gaussian profile is “all core and no wings”, whereas the lorentzian profile is “narrow core and extensive wings”. Thus the onset of the third stage of the curve of growth will occur sooner for a lorentzian profile than for a gaussian profile. We shall find, as we proceed, that for a pure gaussian profile, the third stage of the classic curve of growth is scarcely evident, whereas for a pure lorentzian profile, the second stage is scarcely evident. The classic three-stage curve of growth is exhibited for a Voigt profile in which the gaussian and lorentzian contributions are comparable.

It will be part of the aim of this chapter to predict the curve of growth for gaussian and lorentzian profiles, and also for Voigt profile for different Gauss/Lorentz ratios.

This by itself will be an interesting intellectual exercise – but can it be taken further? Yes it can. Imagine a situation in which we have a spectrum in which the resolution is not sufficient to measure the profiles of individual lines with any great precision. No great imagination is in fact needed, for this will usually be the case with a stellar spectrum. If

we can somehow construct a curve of growth, we might be able to deduce the line profiles, or at least the FWHm (full width at half minimum) of the gaussian component (which will tell us the kinetic temperature) and of the lorentzian component (which will tell us the mean time between collisions and hence the pressure). We cannot, of course, change the column density of a stellar atmosphere. However, some lines of a given element will be weak (because they have a high excitation potential, or a small oscillator strength, or both) and others will be strong (because they have a low excitation potential, or a large oscillator strength, or both), and perhaps a curve of growth can be constructed from many lines of a given atom, and so we will be able to deduce the temperature and the pressure, even though we cannot resolve the details of individual line profiles.

So far, I have discussed the curve of growth for an absorption line. What about an emission line? We understand that the intensity of the centre of an absorption line cannot drop below the “floor” corresponding to zero intensity. But is there a “ceiling” that will stop the growth of the centre of an emission line, or can we go on increasing the intensity of an emission line indefinitely? The answer is that, for a gas in thermodynamic equilibrium, there is indeed a “ceiling”, and that occurs when the radiance per unit wavelength interval of the centre of the emission line reaches the ordinate of the Planck blackbody curve having a blackbody temperature equal to the kinetic and excitation temperature of the gas. When the line centre is completely optically thick, it radiates like a black body at that wavelength. Adding more atoms to the column density will not increase the central intensity. What happens is that photons emitted by atoms near the back of the slab of gas are re-absorbed as they struggle forward towards the front.

## 11.2 *A Review of Some Terms*

Before continuing, a review of some terms such as *absorption coefficient*, *absorptance*, *central depth* and *optical thickness* may be of some use.

Imagine a thin slice of absorbing gas of thickness  $\delta x$ . At one side of the slice, suppose that the specific intensity (radiance) per unit wavelength interval wavelength  $\lambda$  is  $I_\lambda(\lambda)$  and that, after passage of the radiation through the slice, the specific intensity is now only  $I_\lambda + \delta I_\lambda$ . ( $\delta I_\lambda$  is negative.) The fractional decrease in the specific intensity is proportional to  $\delta x$ :

$$-\frac{\delta I_\lambda(\lambda)}{I_\lambda(\lambda)} = \alpha(\lambda)\delta x. \quad 11.2.1$$

$\alpha(\lambda)$  is the *linear absorption coefficient*, and is of dimension  $L^{-1}$ . Now let imagine that, rather than an infinitesimally thin slice of gas, we have a slab of gas of finite thickness  $D$  and that it is sitting in front of a continuum source of specific intensity (radiance) per unit wavelength interval  $I_\lambda(c)$ , where “c” indicates “continuum”. The radiance after passage through the slice is given by integration of equation 11.2.1:

$$I_{\lambda}(\lambda) = I_{\lambda}(c) \exp\left[-\int_0^D \alpha(\lambda) dx\right]. \quad 11.2.2$$

The quantity

$$\tau(\lambda) = \int_0^D \alpha(\lambda) dx \quad 11.2.3$$

is the *optical thickness* of the slab. It is dimensionless. If the slab of gas is *homogeneous*, in the sense that  $\alpha(\lambda)$  is the same throughout the slab and is not a function of  $x$ , this becomes simply

$$\tau(\lambda) = D\alpha(\lambda). \quad 11.2.4$$

Let us suppose that, after passage through the slab of gas, the specific intensity (radiance) of the gas at wavelength  $\lambda$ , which was initially  $I_{\lambda}(c)$ , is now  $I_{\lambda}(\lambda)$ . The fraction of the radiance that has been absorbed is the *absorptance* at wavelength  $\lambda$ ,  $a(\lambda)$ :

$$a(\lambda) = \frac{I_{\lambda}(c) - I_{\lambda}(\lambda)}{I_{\lambda}(c)} \quad 11.2.5$$

It is dimensionless. The relation between absorptance and optical thickness is evidently

$$a(\lambda) = 1 - e^{-\tau(\lambda)}, \quad \tau(\lambda) = -\ln(1 - a(\lambda)). \quad 11.2.6a,b$$

For a gas of very small optical thickness, in which only a tiny fraction of the radiation has been absorbed (which will not in general be the case in this chapter), Maclaurin expansion of either of these equations will show that

$$a(\lambda) \approx \tau(\lambda). \quad 11.2.7$$

If, in addition, the slab of gas is homogeneous,

$$a(\lambda) \approx D\alpha(\lambda). \quad 11.2.8$$

The *absorptance at the line centre* is

$$a(\lambda_0) = \frac{I_{\lambda}(c) - I_{\lambda}(\lambda_0)}{I_{\lambda}(c)}, \quad 11.2.9$$

and we have also called this, in Chapter 10, the *central depth*  $d$ .

The *absorption coefficient* of a gas at the wavelength of an absorption line is proportional to  $n_1$ , the number of atoms per unit volume in the initial (lower) level of the transition.

This is so whether the slab of gas is optically thin or not; we are concerned here with the absorption coefficient and the number density *at a point within the gas*, not with the slab as a whole. The optical thickness of a slab of gas is proportional to  $\mathcal{N}_1$ , the number of atoms per unit area in the line of sight (column density) in the initial level. This is so whether or not the slab is homogeneous. In the sense in which the words *intensive* and *extensive* are used in thermodynamics, it could be said that absorption coefficient is an intensive quantity and optical thickness and absorptance are extensive quantities. While the optical thickness is proportional to  $\mathcal{N}_1$ , the radiance per unit wavelength interval and the equivalent width are not linearly proportional to  $\mathcal{N}_1$  unless the gas is optically thin.

For the terms such as atomic and mass absorption coefficient, opacity, and the distinction between absorption, scattering and extinction, see Chapter 5.

### 11.3 Theory of the Curve of Growth

Let us think again of our homogeneous slab of gas in front of a continuum source. Let  $I_\lambda(c)$  be the radiance per unit wavelength interval of the continuum at wavelength  $\lambda$ . Let  $\tau(x)$  be the optical thickness in the vicinity of a line and  $x = \lambda - \lambda_0$ . If the slab is of thickness  $D$ , the emergent radiance per unit wavelength as a function of wavelength will be

$$I_\lambda(x) = I_\lambda(c) \exp[-\tau(x)]. \quad 11.3.1$$

The equivalent width  $W$  is given by

$$WI_\lambda(c) = \int_{-\infty}^{\infty} (I_\lambda(c) - I_\lambda(x)) dx, \quad 11.3.2$$

or, by making use of equation 11.3.1,

$$W = \int_{-\infty}^{\infty} [1 - \exp\{-\tau(x)\}] dx. \quad 11.3.3$$

If the line is symmetric, this may be evaluated as

$$W = 2 \int_0^{\infty} [1 - \exp\{-\tau(x)\}] dx. \quad 11.3.4$$

In former days, gallant efforts were made to find, using various approximations in the different regimes of the curve of growth, algebraic expressions for evaluating this integral. The availability of modern computers enables us to carry out the integration numerically.

#### 11.4 Curve of Growth for Gaussian Profiles

By “gaussian profile” in the title of this section, I mean profiles that are gaussian in the optically-thin case; as soon as there are departures from optical thinness, there are also departures from the gaussian profile. Alternatively, the *absorption coefficient* and the *optical depth* are gaussian; the *absorptance* is not.

For an optically-thin thermally-broadened line, the optical thickness as a function of wavelength is given by

$$\tau(x) = \tau(0) \exp\left(-\frac{x^2 \ln 2}{g^2}\right), \quad 11.4.1$$

where the HWHM is 
$$g = \frac{V_m \lambda_0 \sqrt{\ln 2}}{c}. \quad 11.4.2$$

Here  $V_m$  is the modal speed. (See Chapter 10 for details.) The line profiles, as calculated from equations 11.3.4 and 11.4.1, are shown in figure XI.2 for the following values of the optical thickness at the line centre:  $\tau(0) = \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8$ .

On combining equations 11.3.4 and 11.4.1 and 2, we obtain the following expression for the equivalent width:

$$W = 2 \int_0^\infty \left(1 - \exp\left\{-\tau(0) e^{-x^2 \ln 2 / g^2}\right\}\right) dx, \quad 11.4.3$$

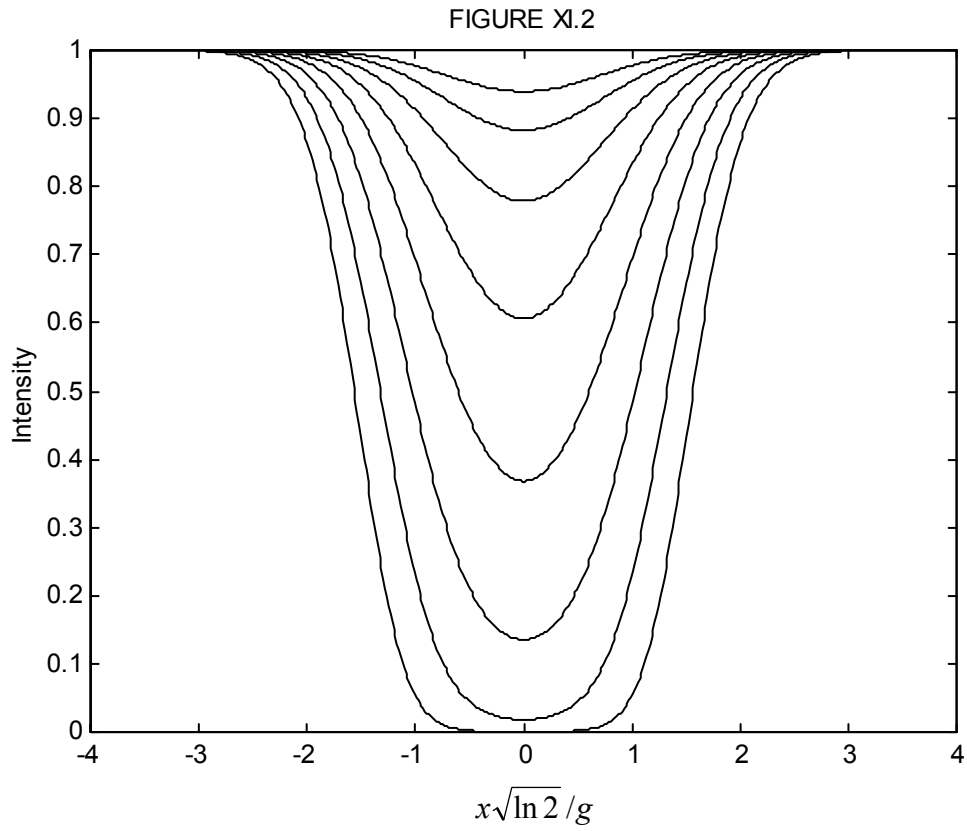
or 
$$W = \frac{2V_m \lambda_0}{c} \int_0^\infty \left(1 - \exp\left\{-\tau(0) e^{-\Delta^2}\right\}\right) d\Delta, \quad 11.4.4$$

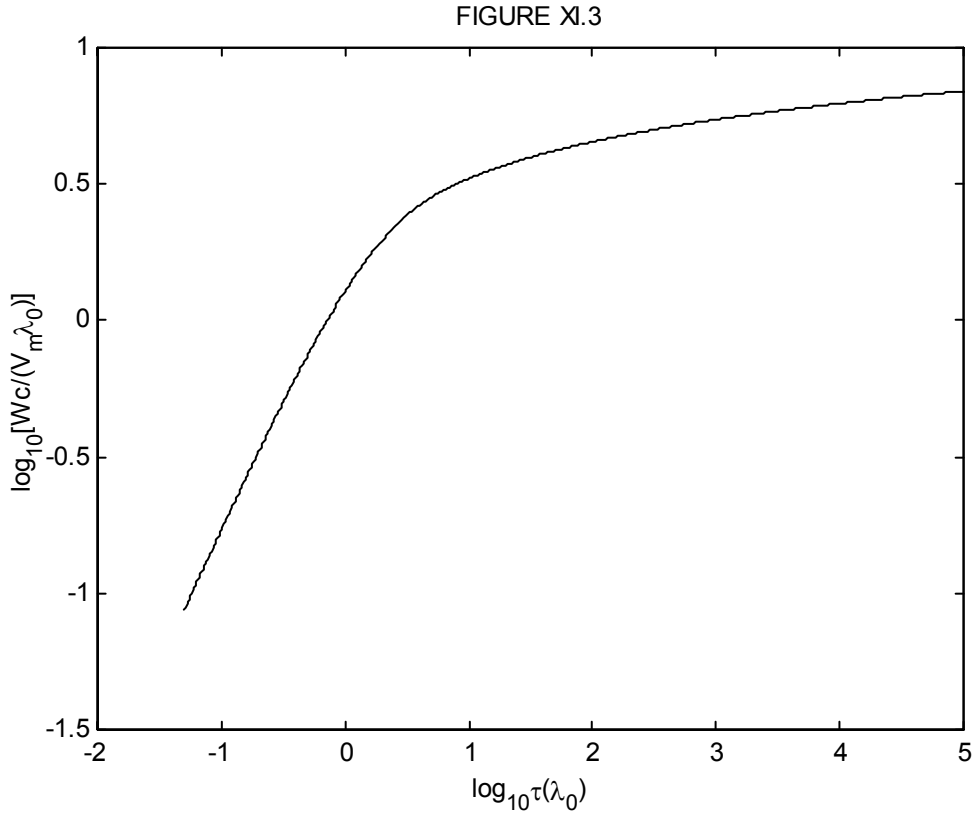
where 
$$\Delta = \frac{x \sqrt{\ln 2}}{g} = \frac{cx}{V_m \lambda_0}. \quad 11.4.5$$

The half-width at half maximum (HWHM) of the expression 11.4.1 for the optical thickness corresponds to  $\Delta = \sqrt{\ln 2} = 0.8326$ . For the purposes of practical numerical integration of equation 11.4.4, I shall integrate from  $\Delta = -5$  to  $+5$  that is to say from  $\pm 6$  times the HWHM. One can appreciate from figure XI.2 that going outside these limits will not contribute significantly to the equivalent width. I shall calculate the equivalent width for central optical depths ranging from  $1/20$  ( $\log \tau(0) = -1.3$ ) to  $10^5$  ( $\log \tau(0) = 5.0$ ).

What we have in figure XI.3 is a curve of growth for thermally broadened lines (or lines that, in the optically thin limit, have a gaussian profile, which could include a thermal and

a microturbulent component). I have plotted this from  $\log \tau(0) = -1.3$  to  $+5.0$ ; that is  $\tau(0) = 0.05$  to  $10^5$ .





### 11.5 Curve of Growth for Lorentzian Profiles

The optical depth of a line broadened by radiation damping is given, as a function of wavelength, by

$$\tau(x) = \tau(0) \frac{l^2}{x^2 + l^2}, \quad 11.5.1$$

where the HWHM is

$$l = \frac{\Gamma \lambda_0^2}{4\pi c} \quad 11.5.2$$

and the optical thickness at the line centre is

$$\tau(0) = \frac{e^2 N_1 f_{12}}{m \epsilon_0 c \Gamma}. \quad 11.5.3$$

Here  $x = \lambda - \lambda_0$ , and the damping constant  $\Gamma$  may include a contribution from pressure broadening. As in section 11.4, in figure XI.4 I draw line profiles for optical thicknesses at the line centre  $\tau(0) = \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8$ .



We see that the wings continue to add to the equivalent width as soon as, and indeed before, the central depth has reached unity. On combining equations 11.3.4, 11.5.1,2 and 3, we obtain the following expression for the equivalent width:

$$W = \frac{\Gamma\lambda_0^2}{2\pi c} \int_0^\infty \left( 1 - \exp\left\{-\frac{\tau(0)}{y^2+1}\right\} \right) dy, \quad 11.5.4$$

in which  $y = x/l$  - i.e. distance from the line centre in units of the HWHM.

If we now substitute  $y = \tan \theta$ , the expression for the equivalent width becomes

$$W = \frac{\Gamma\lambda_0^2}{2\pi c} \int_0^{\pi/2} \frac{1 - \exp[-\tau(0)\cos^2\theta]}{\cos^2\theta} d\theta. \quad 11.5.4$$

Now that we have a finite upper limit, the expression can be integrated numerically without artificial and unjustified truncation. As described in the Appendix to Chapter 10, calculation of the trigonometric function  $\cos$  can be avoided, and hence the integration much speeded up, by the substitution of  $t = \tan(\frac{1}{2}\theta)$ . Although the denominator of the integrand is obviously zero at the upper limit, so is the numerator, and the value of the integrand at the upper limit is finite and equal to  $\tau(0)$ . Figure XI.5 shows the equivalent width, in units of  $\frac{\Gamma\lambda_0^2}{2\pi c}$  as a function of  $\tau(0)$ .

FIGURE XI.4

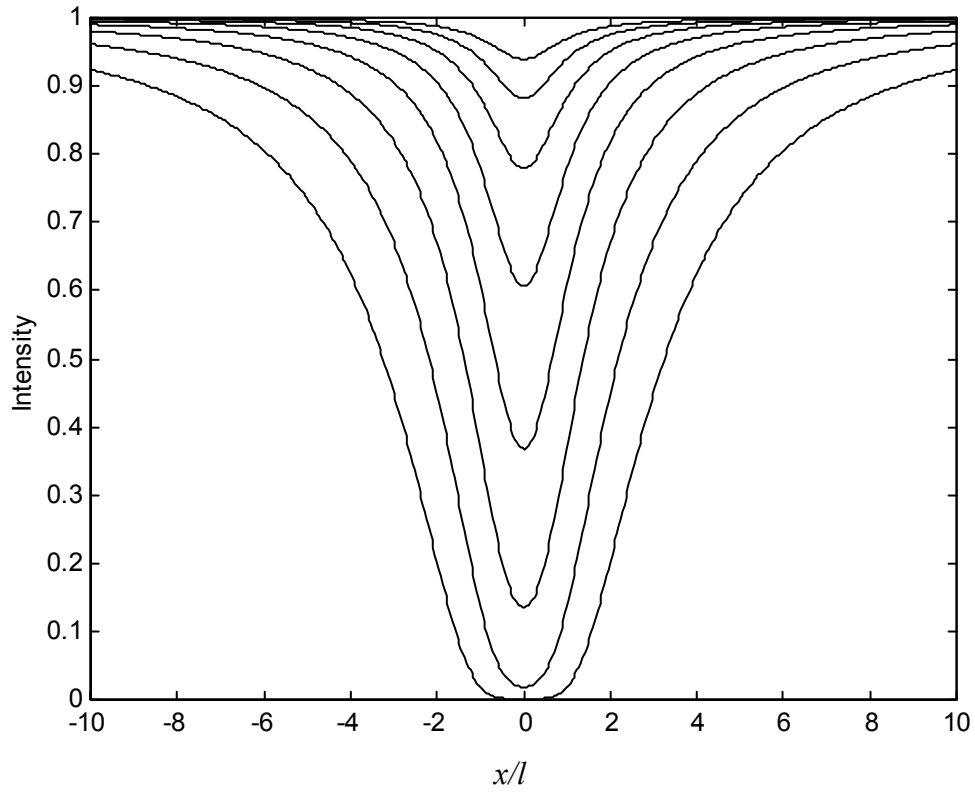
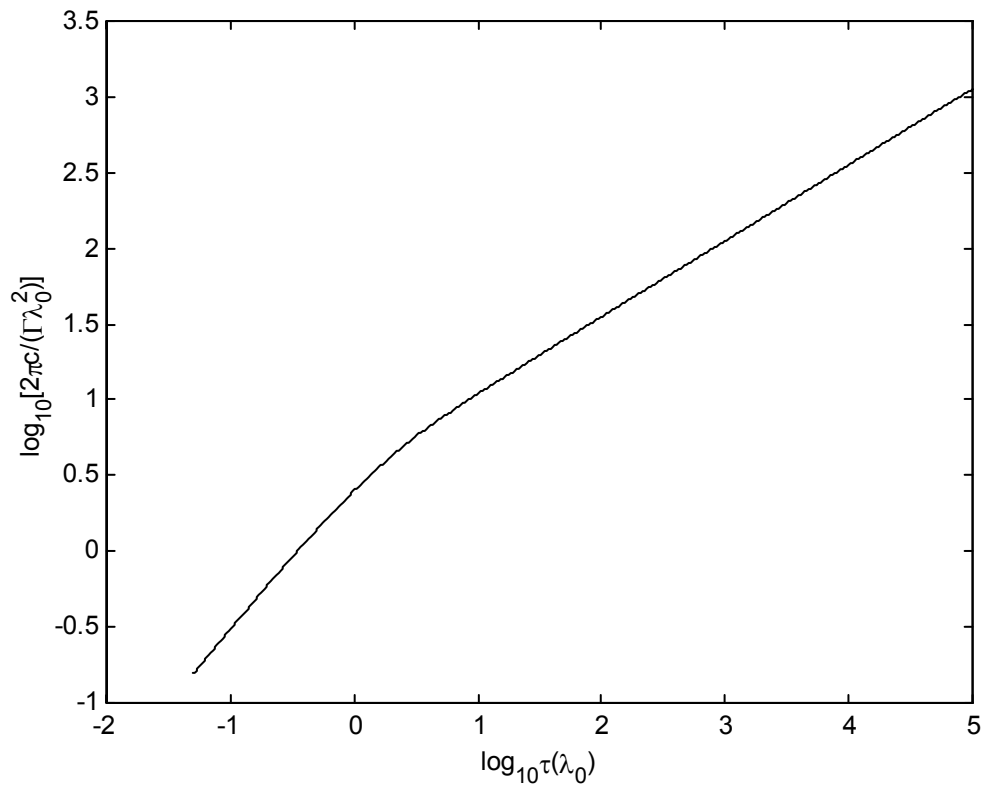


FIGURE XI.5



### 11.6 Curve of Growth for Voigt Profiles

Our next task is to construct curves of growth for Voigt profiles for different values of the ratio of the lorentzian and gaussian HWHMs,  $l/g$ , which is

$$\frac{l}{g} = \frac{\Gamma\lambda_0}{4\pi V_m \sqrt{\ln 2}} = \frac{\Gamma\lambda_0}{V_m \pi \sqrt{\ln 65536}} = \frac{\Gamma\lambda_0}{34.841 V_m}, \quad 11.6.1$$

or, better, for different values of the gaussian ratio  $k_G = \frac{g}{l+g}$ . These should look intermediate in appearance between figures XI.3 and 5.

The expression for the equivalent width in wavelength units is given by equation 11.3.4:

$$W = 2 \int_0^\infty [1 - \exp\{-\tau(x)\}] dx. \quad 11.3.4$$

combined with equation 10.5.20

$$\tau(x) = Cl\tau(0) \int_{-\infty}^\infty \frac{\exp[-(\xi-x)^2 \ln 2/g^2]}{\xi^2 + l^2} d\xi. \quad 10.5.20$$

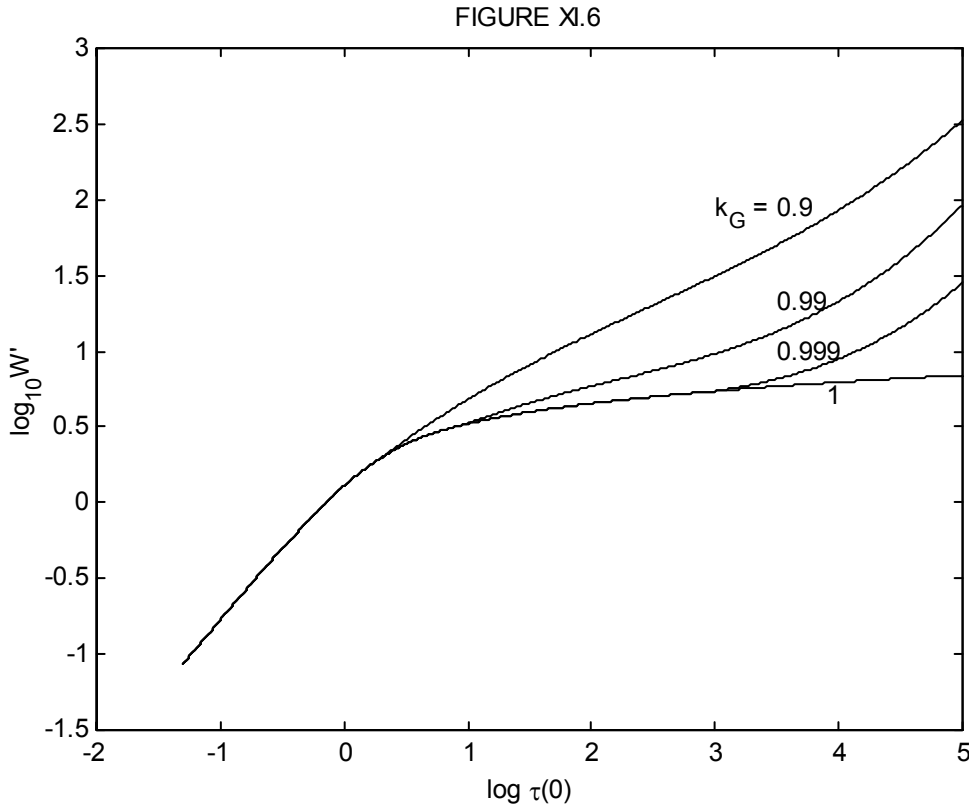
That is:

$$W = 2 \int_0^\infty \left( 1 - \exp \left\{ -Cl\tau(0) \int_{-\infty}^\infty \frac{\exp[-(\xi-x)^2 \ln 2/g^2]}{\xi^2 + l^2} d\xi \right\} \right) dx. \quad 11.6.2$$

Here  $x = \lambda - \lambda_0$ ,  $l$  is the lorentzian HWHM  $= \lambda_0^2 \Gamma / (4\pi c)$  (where  $\Gamma$  may include a pressure-broadening contribution),  $g$  is the gaussian HWHM  $= V_m \lambda_0 \sqrt{\ln 2} / c$  (where  $V_m$  may include a microturbulence contribution), and  $W$  is the equivalent width, all of dimension L. The symbol  $\xi$ , also of dimension L, is a dummy variable, which disappears after the definite integration.  $\tau(0)$  is the optical thickness at the line centre.  $C$  is a dimensionless number given by equation 10.5.23 and tabulated as a function of gaussian fraction in Chapter 10. The reader is urged to check the dimensions of equation 11.6.2 carefully. The integration of equation 11.6.2 is discussed in Appendix A

Our aim is to calculate the equivalent width as a function of  $\tau(0)$  for different values of the gaussian fraction  $k_G = g/(l+g)$ . What we find is as follows. Let  $W' = W\sqrt{\ln 2}/g$ ; that is,  $W'$  is the equivalent width *expressed in units of  $g/\sqrt{\ln 2}$* . For  $\tau(0)$  less than about 5, where the wings contribute relatively little to the equivalent width, we find that  $W'$  is almost independent of the gaussian fraction. The difference in behaviour of the curve of growth for different profiles appears only for large values of  $\tau(0)$ , when the wings assume a larger role. However, for any profile which is less

gaussian than about  $k_G$  equal to about 0.9, the behaviour of the curve of growth (for  $\tau(0) > 5$ ) mimics that for a lorentzian profile. For that reason I have drawn curves of growth in figure XI.6 only for  $k_G = 0.9, 0.99, 0.999$  and 1. This corresponds to  $l/g = 0.1111, 0.0101, 0.0010$  and 0, or to  $\Gamma\lambda_0/V_m = 1.162, 0.1057, 0.0105$  and 0 respectively.



### 11.7 Observational Curve of Growth

Equation 11.6.2 and figure XI.6 show how the equivalent width of a line will grow as the optical thickness is increased, which can be achieved either by increasing the geometrical thickness of the gas under study or by increasing the number density of the absorbers (atoms). Of course, if you are looking at the spectrum of a stellar atmosphere, you cannot do either of these things, but you might be able to construct a curve of growth by looking at many lines from the same element (such as an element from the iron group, which has lots of lines of a wide range of equivalent widths.) Then, by comparing the form of the observed curve of growth with one of the theoretical curves, you could deduce the gaussian and lorentzian HWHm,  $g$  and  $l$ , of your lines (and hence possibly the temperature and pressure) even if your resolution were not sufficient to determine the individual line profiles with any precision. You will notice the word “possibly” – because it has to be remembered that  $g$  includes both a thermal and a microturbulent component, and  $l$  includes both a radiation damping and pressure broadening component.

Provided that the gas under study is homogeneous and of a uniform temperature throughout (I suppose this rules out a stellar atmosphere!)  $g$  should be the same for all lines of a given element. (The microturbulent component of  $g$  should be the same for all elements.) The radiation damping component of  $l$  will vary from line to line, but in practice, at least in the atmosphere of a main sequence star (but not necessarily of a giant, where the atmospheric pressure is much lower), the pressure broadening component of  $l$  will be much greater than the radiation damping component, and hence  $l$  will be the same for all lines of a given element. The theories of thermal, microturbulent, radiation damping and pressure broadening are all covered in Chapter 10.

It is all very well to say plot  $\log W'$  versus  $\log \tau(0)$  for many lines of an iron-group element, but we immediately discover that we don't know either. The dimensionless quantity  $W'$  is given by

$$W' = W\sqrt{\ln 2}/g,$$

and we don't know  $g$  – indeed one of our aims is to find it. The optical depth at the line centre is given by

$$\tau(0) = \frac{N_1 f_{12} e^2}{mc\epsilon_0 \Gamma}, \quad 11.7.2$$

and we don't know  $\Gamma$  – again, one of our aims is to find it. However, these equations can be written in the form

$$\log W' = \log W + \text{constant} \quad 11.7.3$$

$$\text{and} \quad \log \tau(0) = \log N_1 f_{12} + \text{constant}. \quad 11.7.4$$

Also, from Boltzmann's equation (where applicable!), we have

$$\ln N_1 = \ln(N/u) - E_1/(kT) + \ln \varpi_1, \quad 11.7.5$$

$$\text{from which} \quad \log N_1 = \log \varpi_1 + C(E_1) \quad 11.7.6$$

where the “constant” is a function of  $E_1$ , the excitation energy of the lower level of the line. From this, we obtain

$$\log \tau(0) = \log \varpi f + C(E_1). \quad 11.7.8$$

Thus, provided we take a set of lines all having the same lower excitation level (or at least the same lower term, provided that departures from  $LS$ -coupling are not so severe as to scatter the levels widely), we can construct a partial curve of growth by plotting  $\log W$

versus  $\log \varpi f$  for these lines. These will be displaced both vertically and horizontally from the theoretical curves of figure XI.6 by arbitrary amounts, which will not affect the shape of the curve.

We can then take another set of lines, having a different common lower excitation level (or term), and plot another partial curve of growth. It will be displaced horizontally (but not vertically) from the first fragment, and it must be slid horizontally until it meshes in with the first fragment. And so we continue, building up partial curves of growth from sets of lines with a common lower term, sliding them horizontally until they all mesh with each other in a single continuous curve, which we can then compare with the shapes of the theoretical curves to obtain  $g$  and  $l$  and hence the temperature and pressure.

### 11.8 *Interpreting an Optically Thick Profile*

This chapter has been concerned with optically thick lines and with the curve of growth – and how one might recover the parameters  $g$  and  $l$  from the curve of growth even if the profiles of individual lines are not resolved. This section is written rather as an afterthought, albeit an important one, and it does not concern the curve of growth. It discusses how one might analyse the profile of a line that *is* well resolved, but is not optically thin. I use as an example a line that is thermally broadened and which, if optically thin, would have a gaussian profile. The optically thick line is no longer gaussian, but can one recover  $g$  (and hence  $T$ ) from it? Just for a change, and because it is a while since we used the term “source function”, I’ll deal with an emission line.

As discussed in Section 7 of Chapter 5, the radiance of a slab of gas of source function  $S$  is given by

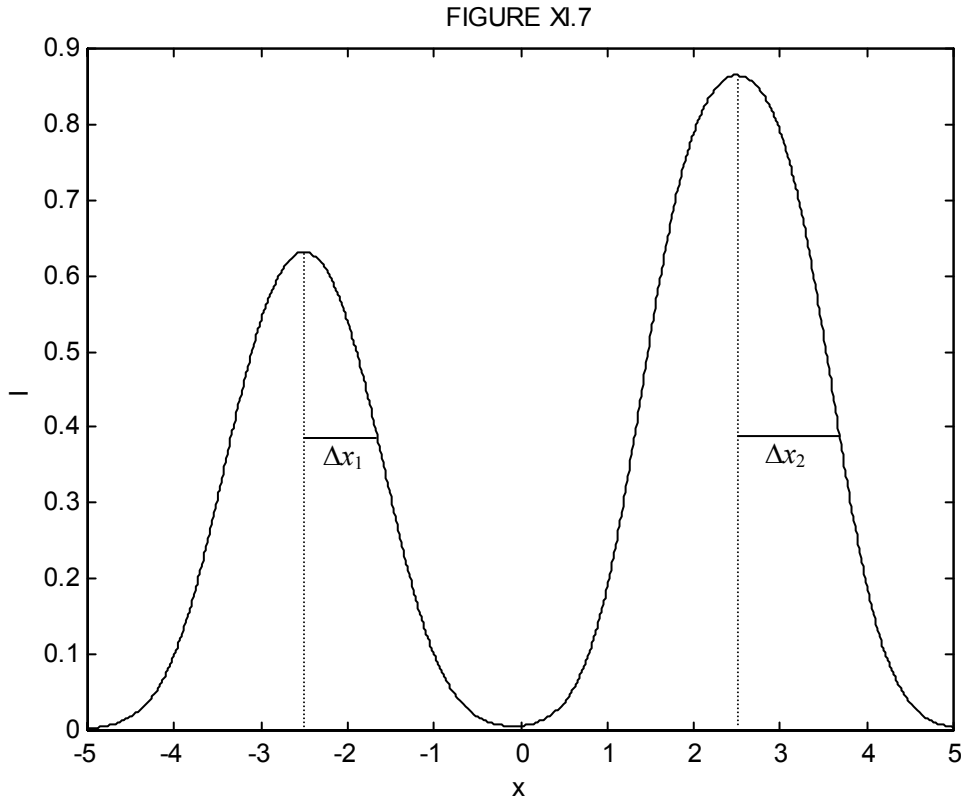
$$I_\lambda(x) = S \int_0^{\tau(x)} e^{-\tau} d\tau = S(1 - e^{-\tau(x)}). \quad 11.8.1$$

Here,  $x = \lambda - \lambda_0$ . For a thermally-broadened line, this becomes

$$I_\lambda(x) = S \left[ 1 - \exp\left\{-\tau(0) \exp(-x^2 \ln 2 / g^2)\right\} \right], \quad 11.8.2$$

where  $g = V_m \lambda_0 / \sqrt{\ln 2}$ .

In figure XI.7, I draw two such profiles, in which the line strength of one is  $A$  times the line strength of the other. The lines, perhaps, belong to the same multiplet and the ratio of the line strengths is known. (I am here using the term *line strength* in the technical sense of Chapter 7.) Figure XI.7 was drawn with  $A = 2$ , but since the lines are not optically thin, neither the ratio of their areas nor the ratio of their heights is two.



We can measure the half widths  $\Delta x_1$  and  $\Delta x_2$  of the two profiles *at the same height*.

Then:

$$S[1 - \exp\{-\tau(0)\exp(-(\Delta x_1)^2 \ln 2/g^2)\}] = S[1 - \exp\{-A\tau(0)\exp(-(\Delta x_2)^2 \ln 2/g^2)\}], \quad 11.8.3$$

after which

$$\frac{(\Delta x_1) \ln 2}{g^2} = \frac{(\Delta x_2) \ln 2}{g^2} - \ln A, \quad 11.8.4$$

and hence

$$g^2 = \frac{[(\Delta x_1)^2 - (\Delta x_2)^2]}{\ln A}. \quad 11.8.5$$

The same technique can be used for lorentzian profiles.

APPENDIX A  
*Evaluation of the Voigt Curve of Growth Integral*

We wish to evaluate the equivalent width as given by equation 11.6.2:

$$W = 2 \int_0^\infty \left( 1 - \exp \left\{ -Cl\tau(0) \int_{-\infty}^\infty \frac{\exp[-(\xi-x)^2 \ln 2/g^2]}{\xi^2 + l^2} d\xi \right\} \right) dx. \quad 11.A.1$$

We slightly simplify it by introducing dimensionless variables  $W'$ ,  $x'$ ,  $\xi'$  and  $l'$  defined by  $X' = \frac{Xc}{\lambda_0 V_m} = \frac{X\sqrt{\ln 2}}{g}$ , where  $X$  is any of  $W$ ,  $x$ ,  $\xi$  or  $l$ . That is, we are expressing lengths in units of  $g/\sqrt{\ln 2}$ . Then:

$$W' = 2 \int_0^\infty \left( 1 - \exp \left\{ -Cl'\tau(0) \int_{-\infty}^\infty \frac{\exp[-(\xi'-x')^2]}{\xi'^2 + l'^2} d\xi' \right\} \right) dx'. \quad 11.A.2$$

We shall eventually want to evaluate the integral as a function of the gaussian ratio  $k_G$ , and we note that  $l' = \left( \frac{1}{k_G} - 1 \right) \sqrt{\ln 2}$ . Since  $\xi'$  is a dummy variable, we could in principle dispense with the prime as superfluous, but it is probably helpful to retain the prime. The limits of the inner integral can be made finite by means of a change of variable. The substitution  $\xi' = l' \tan \theta$  will achieve this, but the introduction of a trigonometric function will increase the computing time. An alternative is  $\xi' = \frac{2l't}{1-t^2}$ , which looks clumsier but is rather faster. The equation becomes

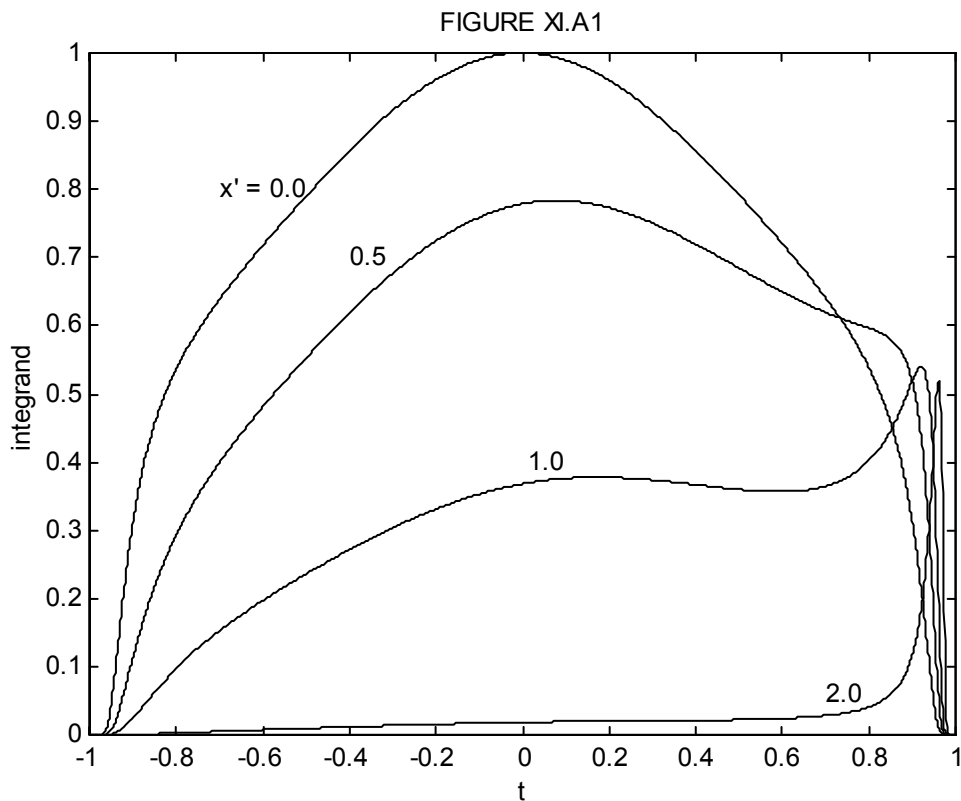
$$W' = 2 \int_0^\infty \left( 1 - \exp \left\{ -2C\tau(0) \int_{-1}^1 \frac{\exp[-(\frac{2l't}{1-t^2} - x')^2]}{1+t^2} dt \right\} \right) dx'. \quad 11.A.3$$

This integral is not particularly easy, and I do not advise merely putting it into a pre-packaged integration program without critical examination. (Indeed I never advocate that!) One is faced with the usual problems of numerical integration: What should be the step size and number of steps? And, in the case of the outer integral, how far should we take the upper limit? One way to determine the necessary number of intervals (and step size) is to try, say 100 intervals, and then try 1000, and see if it makes any difference. Similarly with determining the upper limit to the outer integration, try integrating to  $x' = 5$ , and then to 10, and see if it makes any difference. But this integral has some problems of its own.



Some insight into the problems can be obtained by looking at the behaviour of the inner integrand. This function,  $\frac{\exp[-(\frac{2l't}{1-t^2} - x')^2]}{1+t^2}$ , is shown as a function of  $t$  for four different values of  $x'$  in figure XI.A1. The value of  $l'$  for the figure is  $1/(4\pi)$ , which has no particular significance, since the figure is intended solely for illustrative purposes.

For  $x' = 0$ , the function is symmetric about  $t = 0$ , but not for other values of  $x'$ . For small values of  $x'$ , there is no particular difficulty with the integration, either by Simpson's rule or by seven-point integration (see my Web notes on Celestial Mechanics, Chapter 1).

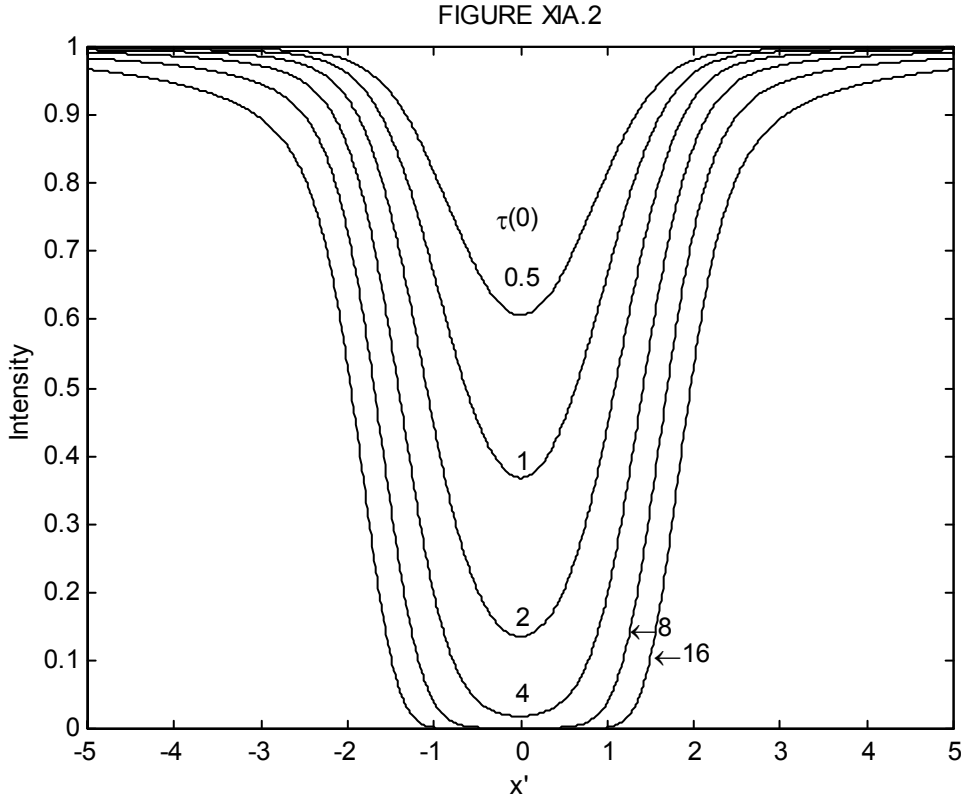


However, for large  $x'$ , the integrand is close to zero for most of the range between  $t = -1.0$  and  $0.0$ , but, close to  $t = 1.0$ , the integrand exhibits a sharp spike, becoming almost a delta-function for larger  $x'$ . A coarse interval for integration may miss the spike altogether, while a very fine interval will use massive amounts of computer time to no good purpose over most of the range of  $t$ . Fortunately, the area under the spike is progressively smaller and smaller for larger and larger  $x'$ , but the nagging question still remains, when may one stop integrating? One useful, efficient and effective technique is to adjust the size of the interval for integration according to the slope of the function. A coarse interval is quite adequate while the graph is near-horizontal, but a much finer integration step kicks in as soon as the slope becomes large. This requires a certain amount of programming legerdemain, but is frequently resorted to in such situations.

The part of equation 11.A 2:

$$\exp \left\{ -Cl\tau(0) \int_{-\infty}^{\infty} \frac{\exp[-(\xi-x)^2 \ln 2/g^2]}{\xi^2 + l'^2} d\xi \right\} \quad 11.A.4$$

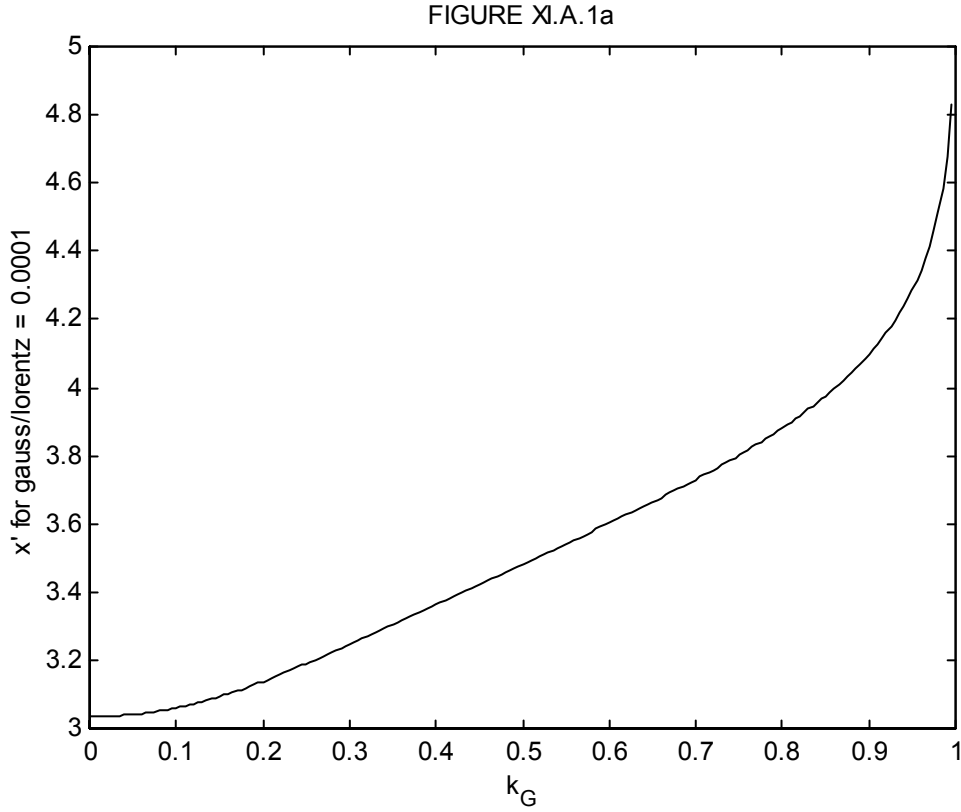
represents the optically thick Voigt profile. This is shown for  $l' = 1/(4\pi)$  for several  $\tau(0)$  in figure XI.A.2. The value  $l' = 1/(4\pi)$  has no particular significance; figure XI.A.2 was drawn solely for illustrative purposes.



The equivalent width (equation 11.A.1) is the area above these curves, and it can be seen again that determining the upper limit for  $x'$  is a problem.

We can, however, remember that the profile that we are integrating is the convolution of a gaussian and a lorentzian, and that  $x'$  is  $\lambda - \lambda_0$  in units of  $2g/\sqrt{\ln 2}$ . The gaussian component has very small wings, whereas the lorentzian profile has extensive wings. The ratio of the gaussian component to the ordinate to the lorentzian component is  $\left( \frac{l'^2 + x'^2}{l'^2} \right) \exp(-x'^2)$ , where  $l' = (1/k_G - 1)\sqrt{\ln 2}$ . The value of  $x'$  at which this ratio drops to 0.0001 is shown as a function of  $k_G$  in figure XI.A.1a. Beyond this point one can treat the wings of the profile as lorentzian.

Thus equation 11.A.2 can be integrated by treating the optically thin profile as a Voigt function up to some  $x' = a$  and as a lorentzian function thereafter. That is, I have written equation 11.A.2 as



$$\begin{aligned}
 W' = & 2 \int_0^a \left( 1 - \exp \left\{ -Cl' \tau(0) \int_{-\infty}^{\infty} \frac{\exp[-(\xi' - x')^2]}{\xi'^2 + l'^2} d\xi' \right\} \right) dx' \\
 & + 2 \int_a^{\infty} \left( 1 - \exp \left\{ -\frac{\tau(0)}{x'^2 + l'^2} \right\} \right) dx'. \quad 11.A.5
 \end{aligned}$$

On substitution of  $\xi' = \frac{2l't}{1-t^2}$  in the first integral and  $x' = l' \tan \theta$  in the second, this becomes this becomes

$$W' = 2 \int_0^a \left( 1 - \exp \left\{ -2C\tau(0) \int_{-1}^1 \frac{\exp[-(\frac{2l't}{1-t^2} - x')^2]}{1+t^2} dt \right\} \right) dx'$$

$$+ 2l' \int_{\alpha}^{\pi/2} \frac{(1 - \exp\{-\tau(0) \cos^2 \theta\})}{\cos^2 \theta} d\theta, \quad 11.A.6$$

where  $\tan \alpha = l'/a$ . The dreaded symbol  $\infty$  has now gone and, further, there is no problem at the upper limit of the second integral, for the value of the integrand when  $\theta = \pi/2$  is unity.