CHAPTER 1
DEFINITIONS OF AND RELATIONS BETWEEN QUANTITIES USED IN RADIATION THEORY

1.1 *Introduction*

An understanding of any discipline must include a familiarity with and understanding of the words used within that discipline, and the theory of radiation is no exception. The theory of radiation includes such words as radiant flux, intensity, irradiance, radiance, exitance, source function and several others, and it is necessary to understand the meanings of these quantities and the relations between them. The meanings of most of the more commonly encountered quantities and the symbols recommended to represent them have been agreed upon and standardized by a number of bodies, including the International Union of Pure and Applied Physics, the International Commission on Radiation Units and Measurement, the American Illuminating Engineering Society, the Royal Society of London and the International Standards Organization. It is rather unfortunate that many astronomers appear not to follow these conventions, and frequent usages of words such as "flux" and "intensity", and the symbols and units used for them, are found in astronomical literature that differ substantially from usage that is standard in most other disciplines within the physical sciences.

In this chapter I use the standard terms, but I point out when necessary where astronomical usage sometimes differs. In particular I shall discuss the astronomical usage of the words "intensity" and "flux" (which differs from standard usage) in sections 1.12 and 1.14 . Standard usage also calls for SI units, although the older CGS units are still to be found in astronomical writings. Except when dealing with electrical units, this usually gives rise to little difficulty to anyone who is aware that 1 watt = 10^7 erg s^{-1}. Where electrical units are concerned, the situation is much less simple.

1.2 *Radiant Flux or Radiant Power, Φ or P.*

This is simply the rate at which energy is radiated from a source, in watts.

It is particularly unfortunate that, even with this most fundamental of concepts, astronomical usage is often different. When describing the radiant power of stars, it is customary for astronomers to use the word *luminosity*, and the symbol *L*. In standard usage, the symbol *L* is generally used for the quantity known as radiance, while in astronomical custom, the word "flux" has yet a different meaning. Particle physicists use the word “luminosity” in yet another quite different sense.

The radiant power ("luminosity") of the Sun is 3.85×10^{26} W.

1.3 *Variation with Frequency or Wavelength*

The radiant flux per unit frequency interval can be denoted by Φ_ν W Hz^{-1}, or per unit wavelength interval by Φ_λ W m^{-1}. The relations between them are

$$\Phi_\lambda = \frac{\nu^2}{c}\Phi_\nu ; \qquad \Phi_\nu = \frac{\lambda^2}{c}\Phi_\lambda \qquad 1.3.1$$

It is useful to use a subscript ν or λ to denote "per unit frequency or wavelength interval", but parentheses, for example $\alpha(\nu)$ or $\alpha(\lambda)$, to denote the value of a quantity at a given frequency or wavelength. In some contexts, where great clarity and precision of meaning are needed, it may not be overkill to use both, the symbol $I_\nu(\nu)$, for example, for the radiant intensity per unit frequency interval at frequency ν.

We shall be defining a number of quantities such as flux, intensity, radiance, etc., and establishing relations between them. In many cases, we shall omit any subscripts, and assume that we are discussing the relevant quantities integrated over all wavelengths. Nevertheless, very often the several relations between the various quantities will be equally valid if the quantities are subscripted with ν or λ.

The same applies to quantities that are weighted according to wavelength-dependent instrumental sensitivities and filters to define a *luminous* flux, which is weighted according to the photopic wavelength sensitivity of a defined standard human eye. The unit of luminous flux is the *lumen*. The number of lumens in a watt of monochromatic radiation depends on the wavelength (it is zero outside the range of sensitivity of the eye!), and for heterochromatic radiation the conversion between lumens and watts requires some careful computation. The number of lumens generated by a lightbulb per watt of power input is called the luminous efficiency of the lightbulb. This may seem at first to be a topic of very remote interest, if any, to astronomers, but those who would observe the faintest and most distant galaxies may well at some time in their careers have occasion to discuss the luminous efficiencies of lighting fixtures in the constant struggle against light pollution of the skies.

The topic of lumens versus watts is a complex and specialist one, and we do not discuss it further here, except for one brief remark. When dealing with visible radiation weighted according to the wavelength sensitivity of the eye, instead of the terms radiant flux, radiant intensity, irradiance and radiance, the corresponding terms that are used become luminous flux (expressed in lumens rather than watts), luminous intensity, illuminance and luminance. Further discussion of these topics can be found in section 1.10 and 1.12.

1.4 *Radiant Intensity, I*

Not all bodies radiate isotropically, and a word is needed to describe how much energy is radiated in different directions. One can imagine, for example, that a rapidly-rotating star might be nonspherical in shape, and will not radiate isotropically. The *intensity* of a source towards a particular direction specified by spherical coordinates (θ , ϕ) is the radiant flux radiated per unit solid angle in that direction. It is expressed in W sr^{-1}, and the standard symbol is *I*. In astronomical custom, the word "intensity" and the symbol *I* are commonly used to describe a very different concept, to which we shall return later.

When dealing with *visible* radiation, we use the phrase *luminous* intensity rather than *radiant* intensity, and the unit is a lumen per steradian, or a candela. At one time, the standard of luminous intensity was taken to be that of a candle of defined design, though the present-day candela (which is one of the fundamental units of the SI system of units) has a different and more precise definition, to be described in section 1.12. The candela and the old standard candle are of roughly the same luminous intensity.

1.5 *"Per unit"*

We have so far on three occasions used the phrase "per unit", as in flux per unit frequency interval, per unit wavelength interval, and per unit solid angle. It may not be out of place to reflect briefly on the meaning of "per unit".

The word *density* in physics is usually defined as "mass per unit volume" and is expressed in kilograms per cubic metre. But do we really mean the mass contained within a volume of a cubic metre? A cubic metre is, after all, a rather large volume, and the density of a substance may well vary greatly from point to point within that volume. Density, in the language of thermodynamics, is an *intensive* quantity, and it is defined *at a point*. What we really mean is the following. If the mass within a volume δV is δm, the *average density* in that volume is $\delta m/\delta V$. The density *at a point* is $\lim_{\delta V \to 0} \frac{\delta m}{\delta V}$, *i.e.* $\frac{dm}{dV}$.

Perhaps the short phrase "per unit mass" does not describe this concept with precision, but it is difficult to find an equally short phrase that does so, and the somewhat loose usage does not usually lead to serious misunderstanding.

Likewise, Φ_λ is described as the flux "per unit wavelength interval", expressed in W m^{-1}. But does it really mean the flux radiated in the absurdly large wavelength interval of a metre? Let $\delta\Phi$ be the flux radiated in a wavelength interval $\delta\lambda$. Then $\Phi_\lambda = \lim_{\delta\lambda \to 0} \frac{\delta\Phi}{\delta\lambda}$; *i.e.* $\frac{d\Phi}{d\lambda}$.

Intensity is flux "per unit solid solid angle", expressed in watts per steradian. Again a steradian is a very large angle. What is actually meant is the following. If $\delta\Phi$ is the flux radiated into an elemental solid angle $\delta\omega$ (which, in spherical coordinates, is $\sin\theta\, \delta\theta\, \delta\phi$) then the *average* intensity over the solid angle $\delta\omega$ is $\delta\Phi/\delta\omega$. The intensity *in a particular direction* (θ , ϕ) is $\lim_{\delta\omega \to 0} \frac{\delta\Phi}{\delta\omega}$. That is, $I = \frac{d\Phi}{d\omega}$.

1.6 *Relation between Flux and Intensity.*

For an isotropic radiator,

$$\Phi = 4\pi I. \qquad 1.6.1$$

For an anisotropic radiator

$$\Phi = \int I d\omega, \qquad 1.6.2$$

the integral to be taken over an entire sphere. Expressed in spherical coordinates, this is

$$\Phi = \int_0^{2\pi}\int_0^{\pi} I(\theta, \phi)\sin\theta d\theta d\phi. \qquad 1.6.3$$

If the intensity is axially symmetric (i.e. does not depend on the azimuthal coordinate ϕ) equation 1.6.3 becomes

$$\Phi = 2\pi\int_0^{\pi} I(\theta)\sin\theta d\theta. \qquad 1.6.4$$

These relations apply equally to subscripted flux and intensity and to luminous flux and luminous intensity.

Example:

Suppose that the intensity of a light bulb varies with direction as

$$I(\theta) = 0.5I(0)(1 + \cos\theta) \qquad 1.6.5$$

(Note the use of parentheses to mean "at angle θ ".)

Draw this (preferably accurately by computer - it is a *cardioid*), and see whether it is reasonable for a light bulb. Note also that, if you put $\theta = 0$ in equation 1.6.5, you get $I(\theta) = I(0)$.

Show that the total radiant flux is related to the forward intensity by

$$\Phi = 2\pi I(0) \qquad 1.6.6$$

and also that the flux radiated between $\theta = 0$ and $\theta = \pi/2$ is

$$\Phi = \tfrac{3}{2}\pi I(0). \qquad 1.6.7$$

1.7 *Absolute Magnitude*

The subject of magnitude scales in astronomy is an extensive one, which is not pursued at length here. It may be useful, however, to see how magnitude is related to flux and intensity. In the standard usage of the word flux, in the sense that we have used it hitherto in this chapter, flux is related to absolute magnitude or to intensity, according to

$$M_2 - M_1 = 2.5 \log (\Phi_1/\Phi_2) \qquad 1.7.1$$

or

$$M_2 - M_1 = 2.5 \log (I_1/I_2) \qquad 1.7.2$$

That is, the difference in magnitudes of two stars is related to the logarithm of the ratio of their radiant fluxes or intensities.

If we elect to define the zero point of the magnitude scale by assigning the magnitude zero to a star of a specified value of its radiant flux in watts or intensity in watts per steradian, equations 1.7.1 and 1.7.2 can be written

$$M = M_0 - 2.5 \log \Phi \qquad 1.7.3$$

or to its intensity by

$$M = M_0' - 2.5 \log I \qquad 1.7.4$$

If by Φ and I we are referring to flux and intensity integrated over all wavelengths, the absolute magnitudes in equations 1.7.1 to 1.7.4 are referred to as absolute *bolometric* magnitudes. Practical difficulties dictate that the setting of the zero points of the various magnitude scales are not quite as straightforward as arbitrarily assigning numerical values to the constants M_0 and M_0' and I do not pursue the subject further here, other than to point out that M_0 and M_0' must be related by

$$M_0' = M_0 - 2.5 \log 4\pi = M_0 - 2.748. \qquad 1.7.5$$

1.8 *Normal Flux Density F*

The rate of passage of energy per unit area normal to the direction of energy flow is the normal flux density, expressed in W m^{-2}.

If a point source of radiation is radiating isotropically, the radiant flux being Φ, the normal flux density at a distance r will be Φ divided by the area of a sphere of radius r. That is

$$F = \Phi/(4\pi r^2) \qquad 1.8.1$$

If the source of radiation is not isotropic (or even if it is) we can express the normal flux density in some direction at distance r in terms of the intensity in that direction:

$$F = I / r^2 \qquad 1.8.2$$

That is, the normal flux density from a point source falls off inversely with the square of the distance.

1.9 *Apparent magnitude*

Although it is not the purpose of this chapter to discuss astronomical magnitude scales in detail, it should be evident that, just as intensity is related to absolute magnitude (both being intrinsic properties of a star, independent of the distance of an observer), so normal flux density is related to apparent magnitude, and they both depend on the distance of observer from star. The relationship is

$$m_2 - m_1 = 2.5 \log (F_1 / F_2) \qquad 1.9.1$$

We could in principle set the zero point of the scale by writing

$$m = m_0 - 2.5 \log F \qquad 1.9.2$$

and assigning a numerical value to m_0, so that there would then be a one-to-one correspondence between normal flux density in W m^{-2} and apparent magnitude. If we are dealing with normal flux density integrated over all wavelengths, the corresponding magnitude is called the apparent *bolometric* magnitude.

1.10 *Irradiance E*

Suppose that some surface is being irradiated from a point source of radiation of intensity I W sr^{-1} at a distance r. The normal flux density ("normal" meaning normal to the direction of propagation), as we have seen, is I / r^2. If the surface being irradiated is inclined so that its normal is inclined at an angle θ to the line joining it to the point source of radiation, the rate at which radiant energy is falling on unit area of the surface will be $I \cos \theta / r^2$.

In any case, the rate at which radiant energy is falling upon unit area of a surface is called the *irradiance* of that surface. It is denoted by the symbol E, and the units are W m^{-2}. In the simple geometry that we have described, the relation between the intensity of the source and the irradiance of the surface is

$$E = (I \cos \theta) / r^2 \qquad 1.10.1$$

If we are dealing with visible radiation, the number of lumens falling per unit area on a plane surface is called the *illuminance*, and is expressed in lumens per square metre, or *lux*. Recall that a lumen is the SI unit of luminous flux, and the candela is the unit of luminous intensity, and that an isotropic point source of light radiating with a luminous intensity of I cd (that is, I lm sr^{-1}) emits a total luminous flux of 4π lm. The relation between the illuminance of a surface and the luminous intensity of a source of light is the same as the relation between irradiance and radiant intensity, namely, equation 1.10.1, or, if the surface is being illuminated normally, equation 1.8.2. If the luminous intensity of a source of light in some direction is one candela, the irradiance of a point on a surface that is closest to the source is 1 lm m^{-2} if the distance is one metre, 1 lm cm^{-2} if the distance is one cm, and 1 lm ft^{-2} if the distance is one foot. A lumen per square metre is a *lux*, and a lumen per square cm is a *phot*. A lumen per square foot is often (usually!) given the extraordinary name of a "foot-candle". This is a most illogical misuse of language, and is mentioned here only because the term is still in frequent use in non-scientific circles. Lumen, candela and lux are, respectively, the SI units of luminous flux, luminous intensity and illuminance. Phot and "foot-candle" are non-SI units of illuminance. The exact definition of the candela will be given in section 1.12; the lumen and lux are derived from the candela. Those who are curious about other strange-sounding units encountered in the quantitative measurement of the visible portion of radiation will also find the definition of "stilb" in section 1.12.

Problem

A table is being illuminated by a light bulb fixed at a distance h vertically above the table. The fixture is such that the socket is above the bulb, and the luminous intensity of the bulb varies as

$$I(\theta) = \tfrac{1}{2} I(0)(1 + \cos\theta) \qquad 1.10.2$$

where θ is the angle from the downward vertical from the bulb. Show that the illuminance at a point on the table at a distance r from the sub-bulb point is

$$E(r) = \frac{I(0)}{2h^2}\left[\frac{(1+r'^2)^{\frac{1}{2}}+1}{(1+r'^2)^2}\right] \qquad 1.10.3$$

where $r' = r/h$, and draw a graph of this for $r' = 0$ to $r' = 2$. For what value of r' does the irradiance fall to half of the sub-bulb irradiance?

Problem

If the table in the above problem is a circular table of radius a, show that the flux that it intercepts is

$$\Phi = \pi I(0)\left[\frac{3a^2 + 2h^2 - 2h(a^2 + h^2)^{\frac{1}{2}}}{2(a^2 + h^2)}\right] \qquad 1.10.4$$

What is this if $a = 0$ and if a ® ∞ ? Is this what you would expect? (Compare equation 1.6.7.)

1.11 *Exitance M*

The *exitance* of an extended surface is the rate at which it is radiating energy (in all directions) per unit area. The usual symbol is M and the units are W m^{-2}. It is an intrinsic property of the radiating surface and is not dependent on the position of an observer.

Most readers will be aware that some property of a black body is equal to σT^4. Technically it is the exitance (integrated over all wavelengths, with no subscript on the M) that is equal to σT^4, so that, in our notation, the Stefan-Boltzmann law would be written

$$M = \sigma T^4, \qquad 1.11.1$$

where σ has the value 5.7×10^{-8} W m^{-2} K^{-4}.

Likewise the familiar Planck equation for a black body:

$$M_\lambda = \frac{2\pi hc^2}{\lambda^5\left(e^{hc/kT} - 1\right)} \qquad 1.11.2$$

gives the exitance per unit wavelength interval.

The word "emittance" is an older word for what is now called exitance.

The *emissivity* of a radiating surface is the ratio of its exitance at a given wavelength and temperature to the exitance of a black body at that wavelength and temperature.

1.12 *Radiance L*

The concept of exitance does nothing to describe a situation in which the brightness of an extended radiating (or reflecting) surface appears to vary with the direction from which it is viewed. For example, the centre of the solar disc is brighter than the limb (which is viewed at an oblique angle), particularly at shorter wavelengths, and the Moon is much brighter at full phase than at first or last quarter.

There are two concepts we can use to describe the directional properties of an extended radiating surface. I shall call them *radiance L*, and "surface brightness" *B*. I first define them, and then I determine the relationship between them. Please keep in mind the meaning of "per unit", or, as it is written in the next sentence, "from unit".

The *radiance L* of an extended source is the irradiance of an observer *from* unit solid angle of the extended source. It is an intrinsic property of the source and is independent of the distance of any observer. This is because, while irradiance of an observer falls off inversely as the square of the distance, the area included in unit solid angle increases as the square of the distance of the observer. While the radiance does not depend on the *distance* of the observer, it may well depend on the *direction* (θ, ϕ) from which the observer views the surface.

The *surface brightness B* of an extended source is the intensity (i.e. flux emitted *into* unit solid angle) from unit projected area of the source. "Projected" here means projected on a plane that is normal to the line joining the observer to a point on the surface. The solid angle referred to here is subtended at a point on the surface. Like radiance, surface brightness is a property intrinsic to the source and is independent of the distance (but not the direction) to the observer.

These concepts may become clearer as I try to explain the relationship between them. This I shall do by supposing that the surface brightness of a point on the surface is *B* in some direction; and I shall calculate the irradiance of an observer in that direction from unit solid angle around the point.

In figure I.1 I draw an elemental area and the vector $\delta\mathbf{A}$ representing that area. In some direction making an angle with the normal to $\delta\mathbf{A}$, the area projected on a plane at right angles to that direction is $\delta A \cos\theta$. We suppose the surface brightness to be *B*, and, since surface brightness is defined to be intensity per unit projected area, the intensity in the direction of interest is $B\delta A \cos\theta$. The irradiance of an observer at a distance *r* from the elemental area is $\delta E = \delta I/r^2 = B\delta A \cos\theta /r^2$. But $\delta A \cos\theta /r^2$ is the solid angle $\delta\omega$ subtended by the elemental area at the observer. Therefore, by definition, $\delta E / \delta\omega$ is *L*, the radiance. Thus $L = B$. We see, then, that radiance *L* and surface brightness *B* are one and the same thing. Henceforth we can use the one term *radiance* and the one symbol *L* for either, and either definition will suffice to define radiance.

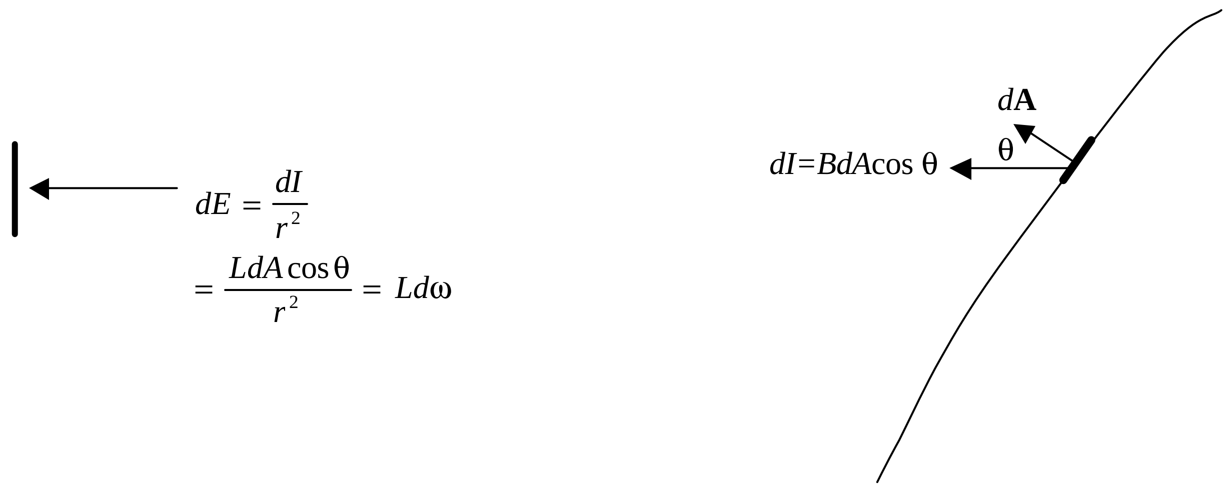


FIGURE I.1

In the figure the surface brightness at some point on a surface in a direction that makes an angle θ with the normal is *B*. The intensity radiated in that direction by an element of area *dA* is $dI = BdA\cos\theta$. The irradiance of a surface at a distance *r* away is $dE = dI/r^2 = BdA\cos\theta / r^2$. But $dA\cos\theta / r^2 = d\omega$, the solid angle subtended by *dA*. But the radiance *L* of a point on the right hand surface is the irradiance of the point in the left hand surface from unit solid angle of the former. Thus $L = B$, and we see that the two definitions, namely surface brightness and radiance, are equivalent, and will henceforth be called just radiance.

Radio astronomers usually use the term "surface brightness". In the literature of stellar atmospheres, however, the term used for radiance is often "specific intensity" or even just "intensity" and the symbol used is *I*. This is clearly a quite different usage of the word intensity and the symbol *I* that we have used hitherto. The use of the adjective "specific" does little to help, since in most contexts in physics, the adjective "specific" is understood to mean "per unit mass". It is obviously of great importance, in both reading and writing on the subject of stellar atmospheres, to be very clear as to the meaning intended by such terms as "intensity".

The radiance per unit frequency interval *of a black body* is often given the symbol B_ν, and the radiance per unit wavelength interval is given the symbol B_λ. We shall see later that these are related to the blackbody exitance functions (see equation 1.11.2 for M_λ) by $M_\nu = \pi B_\nu$ and $M_\lambda = \pi B_\lambda$. Likewise the integrated (over all wavelengths) radiance of a black body is sometimes written in the form $B = aT^4$. Here $a = \sigma / \pi$, *s* being the Stefan-Boltzmann constant used in equation 1.11.1. (But see also section 1.17.)

Summary so far: The concepts "radiance" and "surface brightness", for which we started by using separate symbols, *L* and *B*, are identical, and the single name radiance and the single symbol L suffice, as also will either definition. The symbol B can now be reserved specifically for the radiance of a black body.

Although perhaps not of immediate interest to astronomers other than those concerned with light pollution, I now discuss the corresponding terms used when dealing with visible light. Instead of the terms radiant flux, radiant intensity, irradiance and radiance, the terms used are luminous flux, luminous intensity, illuminance and *luminance*. (This is the origin of the symbol L used for luminance and for radiance.) Luminous flux is expressed in *lumens*. Luminous intensity is expressed in lumens per steradian or *candela*. Illuminance is expressed in lumens per square metre, or *lux*. Luminance is expressed in lm m^{-2} sr^{-1}, or lux sr^{-1}, or cd m^{-2} or *nit*. The standard of luminous intensity was at one time the intensity of light from a candle of specified design burning at a specified rate. It has long been replaced by the candela, whose intensity is indeed roughly that of the former standard candle. The candela, when first introduced, was intended to be a unit of luminous intensity, equal approximately in magnitude to that of the former "standard candle", but making no reference to an actual real candle; it was defined such that the luminance of a black body at the temperature of melting platinum (2042 K) was exactly 600,000 cd m^{-2}. Since 1979 we have gone one step further, recognizing that obtaining and measuring the radiation from a black body at the temperature of melting platinum is a matter of some practical difficulty, and the current definition of the candela makes no mention of platinum or of a black body, and the candela is defined in such a manner that if a source of monochromatic radiation of frequency 5.4×10^{14} Hz has a radiant intensity of 1/683 W sr^{-1} in that direction, then the luminous intensity is one candela. The reader may well ask what if the source in not monochromatic, or what if it is monochromatic but of a different frequency? Although it is not the intention here to treat this topic thoroughly, the answer, roughly, is that scientists involved in the field have prepared a table of a standard "photopic" relative sensitivity of a "standard" photopic human eye, normalized to unity at its maximum sensitivity at 5.4×10^{14} Hz (about 555 nm). For the conversion between watts and lumens for monochromatic light of wavelength other than 555 nm, one must multiply the conversion at 555 nm by the tabular value of the sensitivity at the wavelength in question. To calculate the luminance of a heterochromatic source, it is necessary to integrate over all wavelengths the product of the radiance per unit wavelength interval times the tabular value of the photopic sensitivity curve.

We have mentioned the word "photopic". The retina of the eye has two types of receptor cells, known, presumably from their shape, as "rods" and "cones". At high levels of illuminance, the cones predominate, but at low levels, the cones are quite insensitive, and the rods predominate. The sensitivity curve of the cones is called the "photopic" sensitivity, and that of the rods (which peaks at a shorter wavelengths than the cones) is the "scotopic" sensitivity. It is the standard photopic curve that defines the conversion between radiance and luminance.

I make one last remark on this topic. Namely , together with the metre, kilogram, second, kelvin, ampère and mole, the *candela* is one of the fundamental base units of the Système International des Unités. The *lumen*, *lux* and *nit* are also SI units, but the phot is not. The SI unit of luminance is the *nit*, although in practice this word is rarely heard (lux sr^{-1} or cd m^{-2} serve). The non-SI unit known as the *stilb* is a luminance of one candela per square centimetre.

1.13 *Lambertian Surface*.

A *lambertian* radiating surface [Johann Heinrich Lambert 1728 - 1777] is one whose intensity varies with angle according to Lambert's Law;

$$I(\theta) = I(0)\cos\theta. \qquad 1.13.1$$

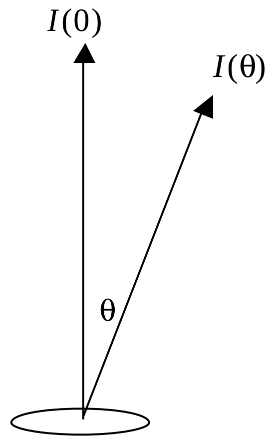


FIGURE I.2

Consider a small element δA of a lambertian radiating surface, such that the intensity radiated by this element in the normal direction is $I(0)$, and the normal radiance is therefore $I(\theta)/\delta A$. The radiance at angle θ is the intensity divided by the projected area:

$$\frac{I(0)\cos\theta}{\delta A\cos\theta} = \frac{I(0)}{\delta A} \qquad 1.13.2$$

Thus the radiance of a lambertian radiating surface is independent of the angle from which it is viewed. Lambertian surfaces radiate isotropically. The radiance of a black body is lambertian. The Sun exhibits limb-darkening; the Sun is not a black body, nor is it lambertian.

For a *reflecting* surface to be lambertian, it is required that the radiance be independent not only of the angle from which it is viewed, but also of the angle from which it is irradiated (or illuminated). In discussing the properties of reflecting surfaces, one often distinguishes between two extreme cases. At the one hand is the perfectly diffusing lambertian surface; blotting paper is sometimes cited as a near lambertian example. The other extreme is the perfectly reflecting surface, or specular reflection (Latin *speculum*, a mirror), in which the angle of reflection equals the angle of incidence. It might be noted that expensive textbooks are often printed on specularly reflecting paper and are difficult to read, whereas inexpensive textbooks are often printed on paper that is approximately lambertian and are consequently easy to read.

The full description of the reflecting properties of a surface requires a bidirectional reflectance distribution function, which is a function of the direction (θ_i, ϕ_i) of the incident light and the direction of the reflected (scattered) (θ_r, ϕ_r) light. Also included in the theory are the several albedoes (normal, geometric and Bond). These concepts are of great importance in the study of planetary physics, but are not pursued further here. Some further details may be found, for example, in Lester, P. L., McCall, M. L. and Tatum, J. B., *J. Roy. Astron. Soc. Can.*, **73**, 233 (1979).

1.14 *Relations between Flux, Intensity, Exitance, Irradiance.*

In this section I am going to ask, and answer, three questions.

i. (See figure I.3)

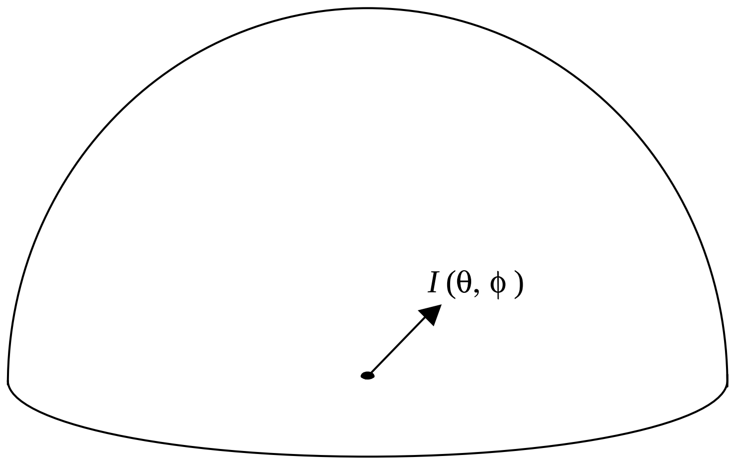


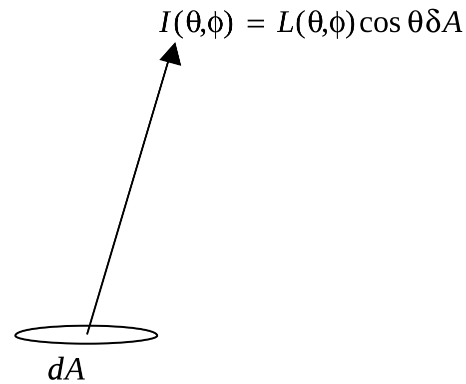
FIGURE I.3

A point source of light has an intensity that varies with direction as $I(\theta, \phi)$. What is the radiant flux radiated into the hemisphere $\theta < \pi/2$? This is easy; we already answered it for a complete sphere in equation 1.6.3. For a hemisphere, the answer is

$$\Phi = \int_0^{2\pi} \int_0^{\pi/2} I(\theta,\phi) \sin\theta \, d\theta \, d\phi. \qquad 1.14.1$$

ii. At a certain point on an extended plane radiating surface, the radiance is $L(\theta, \phi)$. What is the emergent exitance M at that point?

FIGURE I.4



Consider an elemental area dA (see figure I.4). The intensity $I(\theta, \phi)$ radiated in the direction (θ, ϕ) is the radiance times the projected area $\cos\theta\ \delta A$. Therefore the radiant power or flux radiated by the element into the hemisphere is

$$\delta\Phi = \int_0^{2\pi}\int_0^{\pi/2} L(\theta,\phi)\cos\theta\sin\theta d\theta d\phi\delta A, \qquad 1.14.2$$

and therefore the exitance is

$$M = \int_0^{2\pi}\int_0^{\pi/2} L(\theta,\phi)\cos\theta\sin\theta d\theta d\phi. \qquad 1.14.3$$

iii. A point O is at the centre of the base of a hollow radiating hemisphere whose radiance in the direction (θ, ϕ) is $L(\theta, \phi)$. What is the irradiance at that point O ? (See figure I.5.)

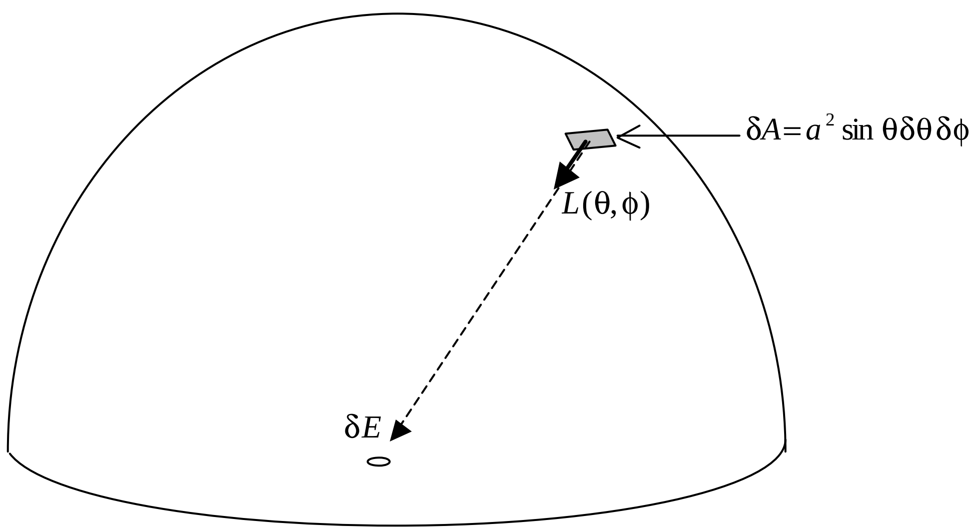


FIGURE I.5

Consider an elemental area $a^2 \sin\theta\,\delta\theta\,\delta\phi$ on the inside of the hemisphere at a point where the radiance is $L(\theta,\phi)$ (figure I.5). The intensity radiated towards O is the radiance times the area:

$$\delta I(\theta,\phi) = L(\theta,\phi)a^2 \sin\theta\delta\theta\delta\phi \qquad 1.14.4$$

The irradiance at O from this elemental area is (see equation (1.10.1)

$$\delta E = \frac{\delta I(\theta,\phi)\cos\theta}{a^2} = L(\theta,\phi)\cos\theta\sin\theta\,\delta\theta\delta\phi, \qquad 1.14.5$$

and so the irradiance at O from the entire hemisphere is

$$E = \int_0^{2\pi}\int_0^{\pi/2} L(\theta,\phi)\cos\theta\sin\theta\,\delta\theta\delta\phi. \qquad 1.14.6$$

The same would apply for any shape of inverted bowl - or even an infinite plane radiating surface (see figure I.6.)

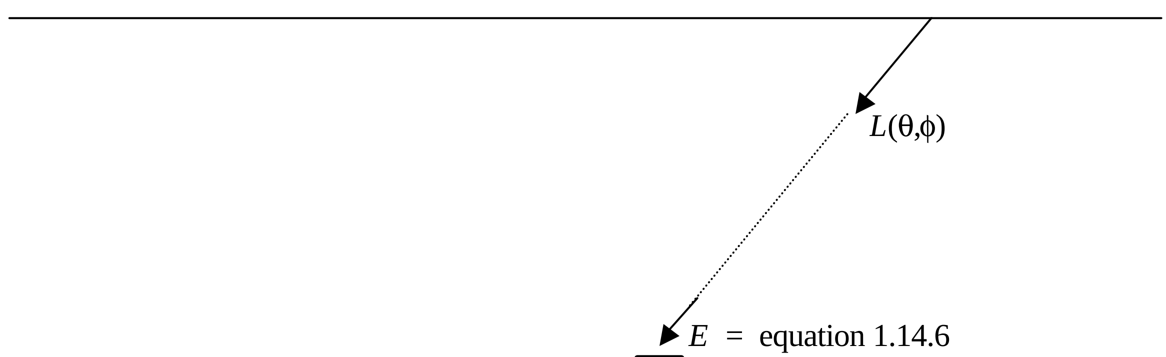


FIGURE I .6

1.15 $A = \pi B$.

There are several occasions in radiation theory in which one quantity is equal to π times another, the two quantities being related by an equation of the form $A = \pi B$. I can think of three, and they are all related to the three questions asked and answered in section 1.14.

If the source in question i of section 1.14 is an element of a lambertian surface, then $I(\theta,\phi)$ is given by equation 1.13.1, and in that case equation 1.14.1 becomes

$$\Phi = \pi I(0) \qquad 1.15.1$$

If the element δA in question ii is lambertian, L is independent of θ and ϕ , and equation 1.14.3 becomes

$$M = \pi L \qquad 1.15.2$$

This, then is the very important relation between the exitance and the radiance of a lambertian surface. It is easy to remember which way round it is if you think of the units in which M and L are expressed and think of π as a solid angle.

If the hemisphere of question iii is uniformly lambertian (for example, if the sky is uniformly dull and cloudy) then L is the same everywhere in the sky, and the irradiance is

$$E = \pi L \qquad 1.15.3$$

1.16 *Radiation Density u*

This is merely the radiation energy density per unit volume, expressed in J m^{-3}, and usually given the symbol u.

1.17 *Radiation Density and Irradiance*

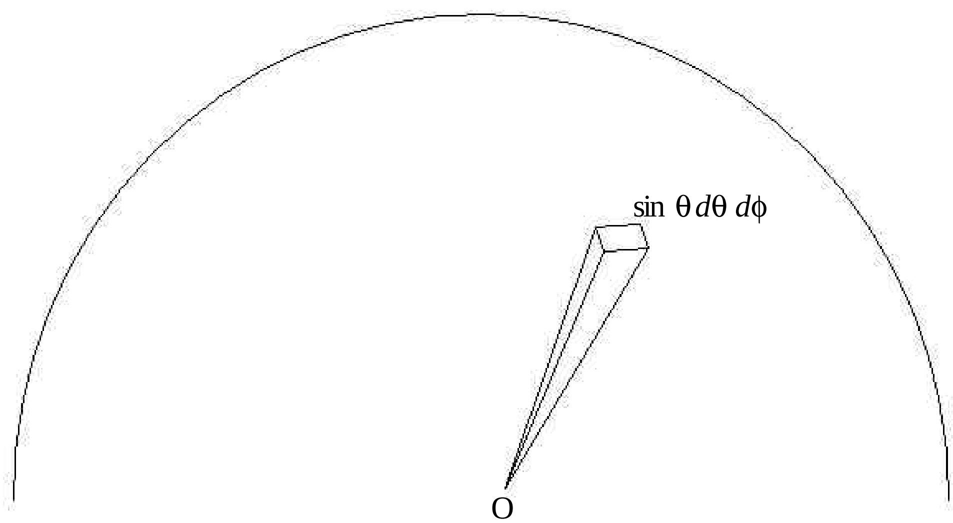


FIGURE I.7

Figure I.7 shows a hemisphere filled with radiation, the energy density being u J m^{-3}. The motion of the photons is presumed to be isotropic, and all are moving at speed c. The centre of the base of the hemisphere, O, is being irradiated - i.e. bombarded with photons coming from all directions. How is the irradiance E at O related to the energy density? How much energy per unit area is arriving at O per unit time? I shall show that the answer is

$$E = uc/4. \qquad 1.17.1$$

Photons are arriving at O from all directions, but only a fraction

$$\frac{\sin\theta\,\delta\theta\,\delta\phi}{4\pi} \qquad 1.17.2$$

are coming from directions within the elemental solid angle between θ and $\theta + \delta\theta$ and ϕ and $\phi + \delta\phi$.

If there are n photons per unit volume, the rate at which photons (moving at speed c) are passing through an elemental area A at O from directions within the elemental solid angle $\sin\theta\,\delta\theta\delta\phi$ is

$$\frac{\sin\theta\,\delta\theta\,\delta\phi}{4\pi} \times nc\delta A\cos\theta\,. \qquad 1.17.3$$

(This is because the elemental area presents a projected area $\delta A\,\cos\theta$ to the photons arriving from that particular direction.)

The rate at which photons arrive per unit area (divide by δA) from the entire hemisphere above (integrate) is

$$\int_0^{2\pi}\int_0^{\pi/2} \frac{nc\sin\theta\cos\theta d\theta d\phi}{4\pi} \qquad 1.17.4$$

The rate at which energy arrives per unit area from the hemisphere above is

$$\int_0^{2\pi}\int_0^{\pi/2} \frac{uc\sin\theta\cos\theta d\theta d\phi}{4\pi} = uc/4, \qquad 1.17.5$$

which was to be demonstrated (*Quod Erat Demonstrandum*)

A more careful argument should convince the reader that it does not matter if all the photons do not carry the same energy. Just divide the population of photons into groups having different energies. The total energy is the sum of the energies of all the photons, whether these are equal or not.

I mentioned in section 1.12 that Stefan's law is sometimes written $L = aT^4$, where a is Stefan's constant divided by π. It is also sometimes written $u = a\,T^4$, and in this case $a = 4s\,/c$.

1.18 *Radiation Pressure P*

Photons carry momentum h/λ and hence exert pressure. Pressure is rate of change of momentum (i.e. force) per unit area.

The pressure P exerted by radiation (in N m^{-2}, or Pa) is related to the energy density u of radiation (in J m^{-3}) by

$$P = 2u \qquad 1.18.1$$

$$P = u \qquad 1.18.2$$

$$P = u/3 \qquad 1.18.3$$

or

$$P = u/6 \qquad 1.18.4$$

depending on the circumstances!

First, we may imagine a parallel beam of photons that have come a long way from their original source. For example, they might be photons that have arrived at a comet from the Sun, and they are about to push material out from the comet to form the tail of the comet. Each of them is travelling with speed c. We suppose that there are n of them per unit volume, and therefore the number of them per unit area arriving per unit time is nc. Each of them carries momentum h/λ. [As in section 1.17 they need not all carry the same momentum. The total momentum is the sum of each.] The rate of arrival of momentum per unit area is $nhc/\lambda = nh\nu$. But $h\nu$ is the energy of each photon, so the rate of arrival of momentum per unit area is equal to the energy density. (Verify that these are dimensionally similar.) If all the photons stick (i.e. if they are absorbed), the rate of change of momentum per unit area (i.e. the pressure) is just equal to the energy density (equation 1.18.2); but if they are reflected elastically, the rate of change of momentum per unit area is twice the energy density (equation 1.18.1).

If the radiation is isotropic, the situation is different. The radiation may be approximately isotropic deep in the atmosphere of a star, though I fancy not completely isotropic, because there is sure to be a temperature gradient in the atmosphere. I suppose for the radiation to be truly isotropic, you'd have to go to the very centre of the star.

We'll start from equation 1.17.4, which gives the rate at which photons arrive at a point per unit area. ("at a point per unit area"? This makes sense only if you bear in mind the meaning of "per unit"!) If the energy of each photon is E, the momentum of each is E/c. (This is the relation, from special relativity, between the energy and momentum of a particle of zero rest mass.) However, it is the normal component of the momentum which contributes to the pressure, and the normal component of each photon is $(E \cos \theta)/c$. The rate at which this normal component of momentum arrives per unit area is found by multiplying the integrand in equation 1.17.4 by this. Bearing in mind that *n*E is the energy density *u*, we obtain

$$\frac{u}{4\pi}\int_0^{2\pi}\int_0^{\pi/2}\sin\theta\cos^2\theta\, d\theta d\phi. \qquad 1.18.5$$

The pressure on the surface is the rate at which the normal component of this momentum is changing. If the photons stick, this is

$$\frac{u}{4\pi}\int_0^{2\pi}\int_0^{\pi/2}\sin\theta\cos^2\theta\, d\theta d\phi \;=\; u/6. \qquad 1.18.6$$

But if they bounce, it is twice this, or *u*/3.