CHAPTER 3
Ellipsoids and Hyperboloids
(Central Quadrics)

3.1 *Ellipsoid*

The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad 3.1.1$$

represents an ellipsoid, whose principal semi-axes are of lengths *a*, *b* and *c*. When written in that form, the axes of the ellipsoid are along the axes of coordinates, and the centre of the ellipsoid is at the origin of coordinates. The ellipsoid does indeed have a centre, and it is therefore a *central quadric*. It is symmetric by reflection through the origin. If *x*, *y* , *z* are replaced by $-x, -y, -z,$ the equation is unaltered. The equation does not contain any odd powers of *x*, *y* or *z*.

If $a \neq b \neq c$, it is a *triaxial ellipsoid.*

If $a = b > c$, it is an *oblate spheroid* (like planet Earth - flattened at the poles), formed by rotating an ellipse about its minor axis.

If $a > b = c$, it is a *prolate spheroid* (like a rugger ball), formed by rotating an ellipse about its major axis.

Circular Sections of an Ellipsoid

(This topic is of interest in connection with the propagation of light in an anisotropic crystal.)

Consider the triaxial ellipsoid of equation 3.1.1, with $a > b > c$, cross-sections of which are shown in figure III.1. By moving the eye around the *yz*, *zx* and *xy* planes. we see that there are two circular cross-sections. They are of radius *b*. The normals to these circular cross-sections are in the *zx* plane. We shall show, using figure III.2, that they each make an angle $\psi = \cos^{-1}\left(\frac{a}{b}\sqrt{\frac{b^2 - c^2}{a^2 - c^2}}\right)$ with the *z*-axis, and hence $2\psi = 2\cos^{-1}\left(\frac{a}{b}\sqrt{\frac{b^2 - c^2}{a^2 - c^2}}\right)$ with each other.

The drawings in figure III.1 and III.2 were made with $a = 1$, $b = \frac{2}{3}$, $c = \frac{1}{3}$, so that the normals to the two plane cross-sections make an angle $\cos^{-1}\sqrt{\frac{27}{32}} = 23^{\circ}.3$ with the *z*-axis and 46º.6 with each other.

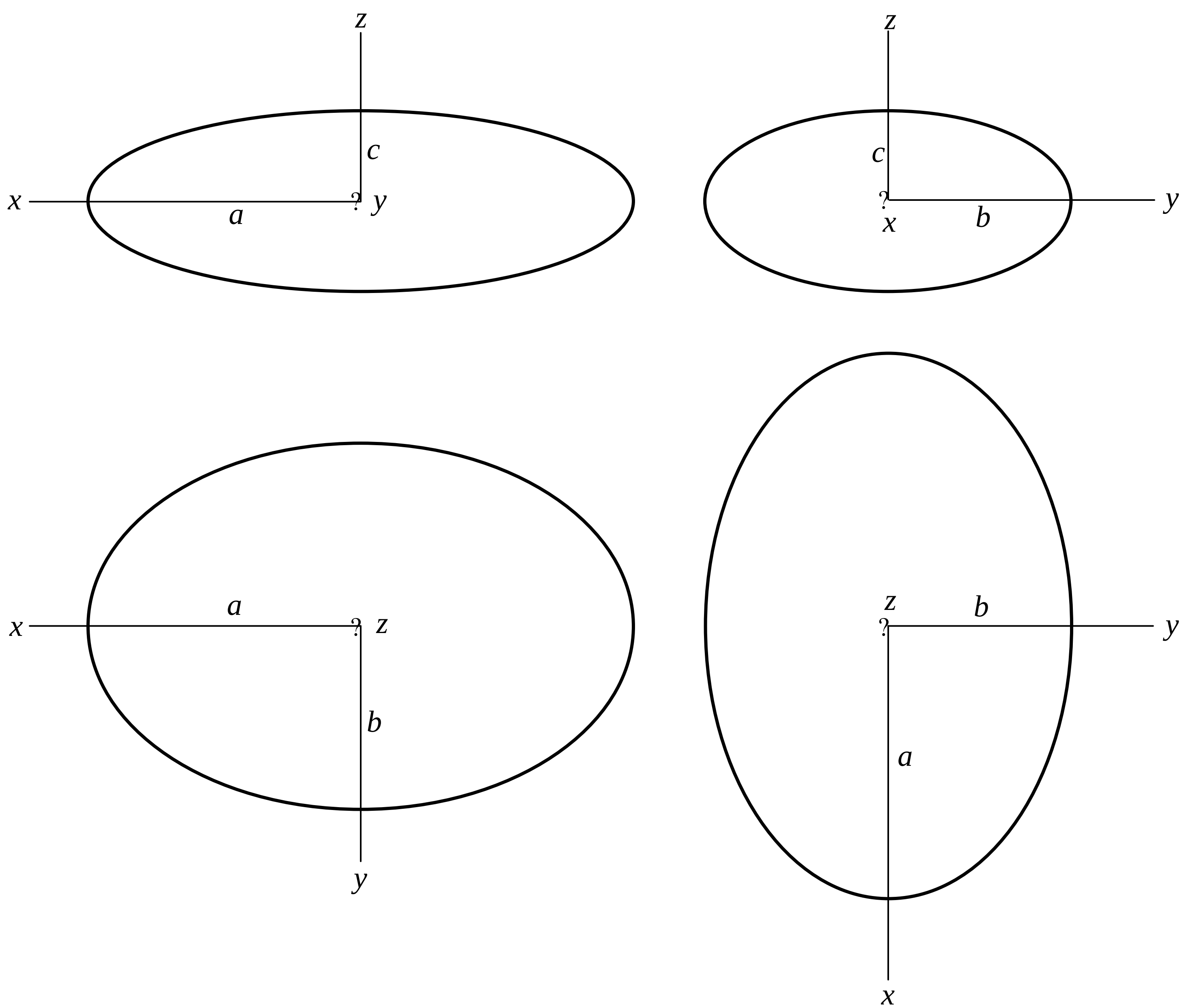

FIGURE III.1

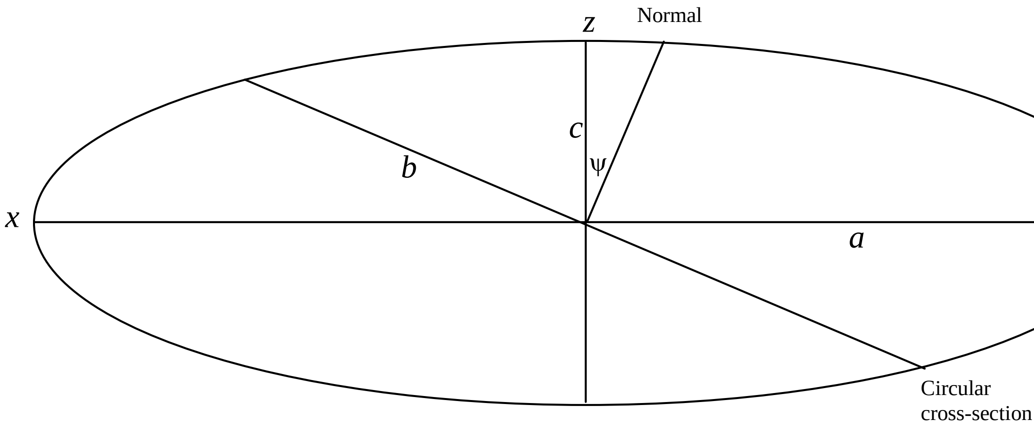

FIGURE III.2

In figure III.2, we show the elliptical cross-section in the *zx*-plane, whose equation is

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1. \qquad 3.1.2$$

We also show the circular cross-section of radius *b*, and its normal, making an angle ψ with the *z*-axis. We see that $x = b\cos\psi$ and $z = b\sin\psi$. On substitution of these into equation 3.1.2, we very soon find that

$$\cos\psi = \frac{a}{b}\sqrt{\frac{b^2 - c^2}{a^2 - c^2}}. \qquad 3.1.3$$

3.2 *Hyperboloid of One Sheet.*

Imagine the hyperbola

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 \qquad 3.2.1$$

The semi transverse axis is of length *a*. (We needn't be concerned in the present context with a geometric meaning for *c*, but it is the length of the semi transverse axis of the conjugate hyperbola.)

I've drawn it in figure III.3 for *a* = 1, *c* = 2.

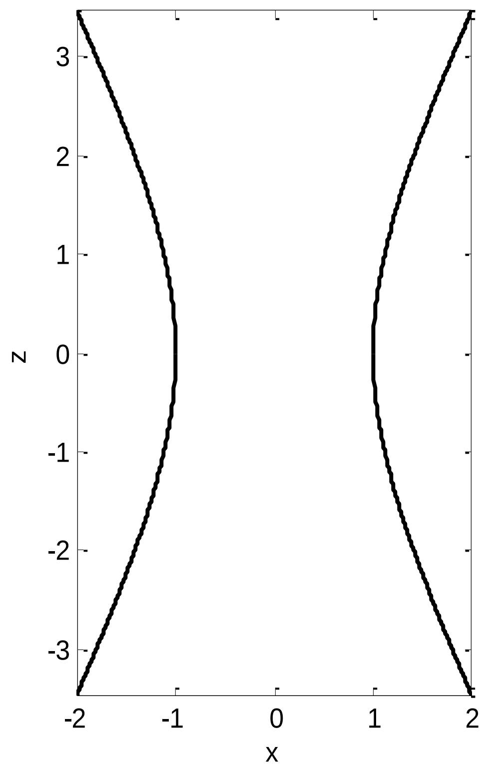

FIGURE III.3

Now rotate the hyperbola about the vertical axis. The figure you get is a *hyperbola of one sheet of circular cross-section*. It somewhat resembles a cooling tower at a power station. Its equation is

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{c^2} = 1 \qquad 3.2.2$$

At a distance z along the z-axis, the cross-section is a circle of radius $a\sqrt{\frac{z^2}{c^2}+1}$.

However, in general a hyperboloid of one sheet need not be of circular cross-section. It can be of elliptical cross-section, with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \qquad 3.2.3$$

although this, of course, cannot be obtained by rotation of a hyperbola..

When written in the form of equation 3.2.3, the three axes of the hyperboloid are along the coordinate axes, and the centre of the hyperboloid is at the origin of coordinates. The hyperboloid does indeed have a centre, and, like the ellipsoid, it is therefore a *central quadric*. It is symmetric by reflection through the origin. If x, y , z are replaced by $-x, -y, -z,$ the equation is unaltered. The equation does not contain any odd powers of x, y or z.

3.3 *Hyperboloid of Two Sheets.*

Now rotate the hyperbola of figure III.3 about the horizontal axis. In this case you get a *hyperboloid of two sheets* of circular cross-section. Its equation is

$$\frac{x^2}{a^2} - \frac{y^2}{c^2} - \frac{z^2}{c^2} = 1. \qquad 3.3.1$$

At a distance x along the x-axis, the cross-section is a circle of radius $c\sqrt{\frac{x^2}{a^2} - 1}$.

At $x = a$, this is zero. For $x < 0$ this is not real.

However, in general a hyperboloid of one sheet need not be of circular cross-section. It can be of elliptical cross-section, with equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \qquad 3.3.2$$

although this, of course, cannot be obtained by rotation of a hyperbola.

Like the ellipsoid and the hyperboloid of one sheet, the hyperboloid of two sheets has no terms in odd powers of the coordinates, and it is a central quadric.

Most references that I have seen say that the primary mirror of a Ritchey-Chrétien telescope is "hyperbolic". Since a hyperbola is a two-dimensional figure, this isn't strictly correct. The shape of the mirror of a Ritchey-Chrétien telescope is actually part of one sheet of a circular hyperboloid of two sheets, formed by rotating a hyperbola about its transverse axis.

Summary:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad \text{ellipsoid}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad \text{hyperboloid of one sheet}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$ hyperboloid of two sheets