Chapter 16
Integration of Functions of a Complex Variable

16.1 *Introduction*

Let $z = x + iy$ be a complex variable, and let $w = f(z) = u + iv$ be a function of z

Integrate $f(z)$ in the z-plane from $a_1 + ib_1$ to $a_2 + ib_2$.

It might be, if you are new to this topic, that not only do you not know how to do it, but you are not entirely certain exactly what it is that you are being asked to do.

Nevertheless, even from this stage of uncertainty, you may be tempted to ask: Does the result depend on how you get from $a_1 + ib_1$ to $a_2 + ib_2$? If you go from one point to the other in a direct straight line, do you get the same result if you go by a more circuitous route?

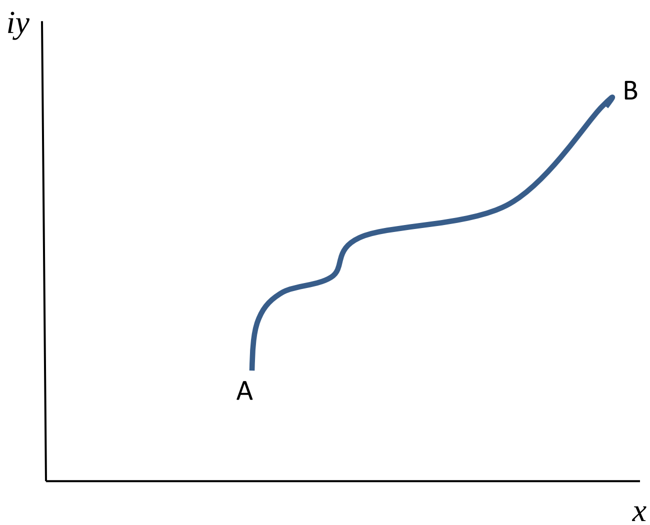
It turns out that this is in fact a very legitimate question. And it turns out that the answer to the question is that the integral is route-independent if $f(z)$ is an analytic function (see Chapter 15 for the meaning of an analytic function and how you can tell whether or not a given function is analytic). But if the function is not analytic, the integral is route-dependent.

You may very well, then, make this speculation: The integral of an analytic function around a closed path must be zero. This, again, is a very legitimate speculation – the truth of which, at this stage, I neither confirm nor deny. I'll discuss it later in the chapter.

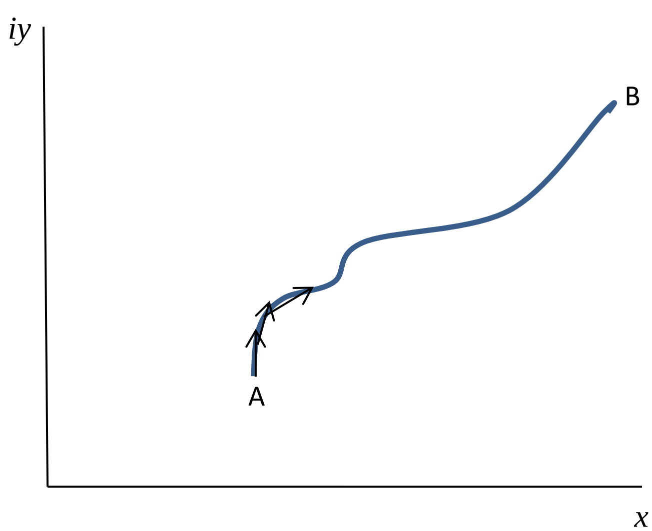
In spite of that, and still from your state of uncertainty, you may well ask another question: Suppose that $f(z)$ is *not* analytic, and the integral around a closed path is consequently not zero, does the integral around the closed path depend on which way (clockwise or counterclockwise) you go round the path? That's yet another legitimate question, and the answer is that the integral does indeed depend on which way you go. The integral around the path clockwise is minus the integral around the path counterclockwise. Therefore, in calculating the integral of a nonanalytic function around a closed path, you must always make it clear whether you are integrating clockwise or counterclockwise.

It is very often customary to calculate such an integral in the counterclockwise direction. This may be often the custom, and some authors may mistakenly believe that the custom is universal, but it in no way releases you from your obligation to make it absolutely clear to whomever may be reading or listening to your work which way you are going.

Integrate $f(z)$ from A to B in the drawing below.



In the redrawing below I have drawn three small arrows, representing three differentials of complex numbers. Although complex numbers are not vectors, they add like vectors, so we shall treat them as vectors in this discussion. We'll call these little vectors δz_1 , δz_2 , δz_3, starting at the points z_0 , z_1 , $z_{2.}$ Let us add all these little arrows, as vectors or as complex numbers, all the way from A to B. The limit of the sum of all the $z_i \delta z_i$ as all the δz_i approach zero is the *contour integral* of $f(z)$ from A to B along the route shown. In general this integral depends on the route taken – unless $f(z)$ is an *analytic* function, in which case the contour integral is route-independent.



16.2 *Example*

By this time we have a vague idea of what is meant by a contour integral. The best way to get a more concrete understanding is actually to carry out a contour integration of a particular function along a particular route.

Thus: Integrate the function $w = f(z) = \sin z$ in a straight line from $1 - i$ to $1 + i$.

Anywhere along this route, the real part of z (Re z for short) is 1, and Im $z = iy$. That is, along this route, $z = 1 + iy$, $dz = idy$, $w = \sin(1 + iy)$.

Thus:

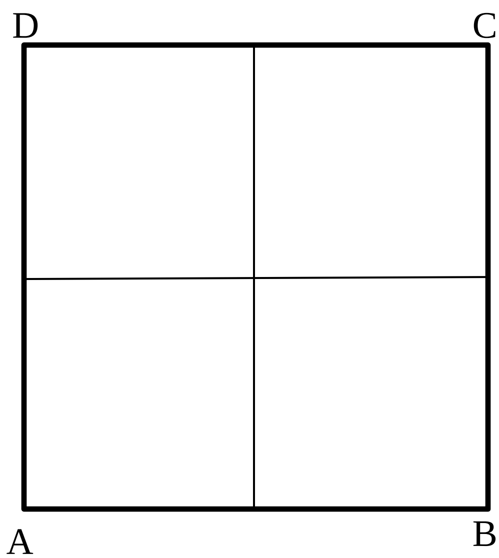
$$\int_{1-i}^{1+i} wdz = \int_{-1}^{1} \sin(1+iy)idy = [\cos(1+iy)]_1^{-1} = \cos(1-i) - \cos(1+i)$$

That is the answer to the problem, but you may wish to go further to make it more understandable

$$\cos(1-i) - \cos(1+i) = 2\sin i \sin 1 = 2i \sinh 1 \sin 1 = 1.9778i$$

16.3 *Integration of analytic functions around closed paths.*

We discovered, by numerical means in Part I that the functions z^2, $\sin z$, e^z appeared to be analytic or very nearly so, and in Part II we were invited to prove that they were analytic by application of the Riemann-Cauchy conditions. That being so, the contour integrals of these functions round a closed path should be zero. Let us verify this by integrating each of them around a square.



The points A, B, C, D represent the following complex numbers:

A: $-1-i$

B: $1-i$

C: $1+i$

D: $-1+i$

Along the sides of the square:

AB: $z = x - i \qquad dz = dx$

BC: $z = 1 + iy \qquad dz = idy$

CD: $z = x + i \qquad dz = dx$

DA: $z = -1 + iy \qquad dz = idy$

First, let us integrate $w = z^2$ counterclockwise around the square, first along AB, then along BC, then along CD and finally along CA.

Along the sides of the square:

AB: $w = z^2 = (x-i)^2 = x^2 - 1 - 2xi$

$$\int_A^B wdz \quad \int_{-1}^{1}(x^2 - 1 - 2xi)\,dx = -\frac{4}{3}$$

BC: $w = z^2 = (1+iy)^2 = 1 - y^2 + 2yi$

$$\int_B^C wdz \quad \int_{-1}^{1}(1 - y^2 + 2yi)idy = \frac{4}{3}i$$

CD: $w = z^2 = (x+i)^2 = x^2 - 1 + 2xi$

$$\int_C^D wdz \quad \int_{1}^{-1}(x^2 - 1 + 2xi)\,dx = \frac{4}{3}$$

DA: $w = z^2 = (-1+iy)^2 = 1 - y^2 - 2yi$

$$\int_D^A wdz \quad \int_{-1}^{1}(1 - y^2 - 2yi)idy = -\frac{4}{3}i$$

Thus, as expected, the contour integral around the square is zero.

Now let's do the same thing with $w = \sin z$

Along the sides of the square:

AB: $w = \sin z = \sin(x - i)$

$$\int_A^B wdz = \int_{-1}^{1} \sin(x - i)\, dx = \cos(-1 - i) - \cos(1 - i)$$

BC: $w = \sin z = \sin(1 + iy)$

$$\int_B^C wdz = \int_{-1}^{1} \sin(1 + iy)\, i\, dy = \cos(1 - i) - \cos(1 + i)$$

CD: $w = \sin z = \sin(x + i)$

$$\int_C^D wdz = \int_{1}^{-1} \sin(x + i)\, dx = \cos(1 + i) - \cos(-1 + i)$$

DA: $w = \sin z = \sin(-1 + iy)$

$$\int_D^A wdz = \int_{1}^{-1} \sin(-1 + iy)\, i\, dy = \cos(-1 + i) - \cos(-1 - i)$$

Thus, as expected, the contour integral around the square is zero.

Now let's do the same thing with $w = e^z$

Along the sides of the square:

AB: $w = e^z = e^{x-i} = e^{-i}e^x$

$$\int_A^B wdz = e^{-i}\int_{-1}^{1} e^x dx = e^{-i}(e - e^{-1}) = (\cos 1 - i \sin 1)\,(e - e^{-1})$$

$$= 1.270 - 1.978i$$

BC: $w = e^z = e^{1+iy} = e.e^{iy}$

$$\int_B^C wdz = ie\int_{-1}^{1} e^{iy}\, dy = e\left(e^i - e^{-i}\right) = 2ie \sin 1 = 4.575i$$

CD: $w = e^z = e^{x+i} = e^i e^x$

$$\int_C^D w dz = e^i \int_1^{-1} e^x dx = e^i(e^{-1} - e) = (\cos 1 + i \sin 1)(e^{-1} - e)$$

$$= -1.270 - 1.978i$$

DA: $w = e^z = e^{-1+iy} = e^{-1}.e^{iy}$

$$\int_D^A w dz = ie^{-1}\int_1^{-1} e^{iy}\, dy = e^{-1}\left(e^{-i} - e^i\right) = 2ie^{-1}\sin 1 = -0.619i$$

The integral around the entire square is $1.270 - 1.270 + (-1.978 + 4.575 - 1.978 - 0.619)i$,

which, as expected, is zero.

It is also easy to integrate the function $w = z^2$ around the unit circle $z = e^{i\theta}$. In this case, $dz = ie^{i\theta}d\theta$. The integral around the unit circle is $\oint w dz = \int_0^{2\pi} e^{2i\theta}.ie^{i\theta}\, d\theta$, which is zero, as expected.

16.3 *Integration of an analytic function between two points*

At the very beginning of Section 16.1 I asked the reader to integrate $f(z)$ in the z-plane from $a_1 + ib_1$ to $a_2 + ib_2$. We now know enough to understand what the question means and, if the function is analytic, how to carry the integration out. If the function is analytic (as determined, for example, by the Cauchy-Riemann conditions), the integral is route-independent, so it will usually be convenient first to integrate with respect to x parallel to the real axis from a_1 to a_2, and then to integrate with respect to y parallel to the imaginary axis from b_1 to b_2.

For example, integrate z^2 from $2 + 3i$ to $4 + 7i$.

First, we integrate with respect to x from $z = 2 + 3i$ to $4 + 3i$. Along this path, $z = x + 3i$ and $dz = dx$ and $z^2 = x^2 - 9 + 6i$. Also where $z = 2 + 3i,\ x = 2$, and where $z = 4 + 3i,\ x = 4$.

Thus $\int_{2+3i}^{4+3i} z^2\, dz = \int_2^4 (x^2 - 9 + 6ix)\, dx = \frac{1}{3}(2 + 108i)$

Next, we integrate with respect to y from $z = 4 + 3i$ to $z = 4 + 7i$. Along this path, $z = 4 + iy$ and $dz = idy$ and $z^2 = 16 - y + 2iy$. Also, where $z = 4 + 3i,\ y = 3$, and where $z = 4 + 7i$, $y = 7$.

Thus $\int_{4+3i}^{4+7i} z^2\,dz \;=\; i\int_3^7 (16 - y^2 + 2iy)dy \;=\; \frac{1}{3}\,(-124 \;+ 120i)$

Therefore $\int_{4+3i}^{4+7i} z^2\,dz \;=\; \frac{1}{3}(-122 \;+\; 228i)$

We can do the same procudure with a nonanalytic function, except that then, of course, the result is valid only for that particular route.

16.4 $w = f(z) = z^{-n}$

We shall try to integrate z^{-n} , where *n* is a positive integer, around the square ABCD of Section 16.3. The function is well-behaved around the perimeter of the square, but there is a singularity *within* the square – namely an infinity at the origin of coordinates. The integral around a closed circuit will not be zero unless the function and its derivative are analytic – *i.e.* finite, single-valued and continuous – *everywhere* on and within the perimeter.

First, we'll try with *n* = 1. That is, we'll integrate z^{-1} around the square.

Along the sides of the square:

AB: $\quad w = \dfrac{1}{x - i} \;=\; \dfrac{x + i}{x^2 \;+\; 1}$

$$\int_{\mathrm{A}}^{\mathrm{B}} w dz = \int_{-1}^{1} \frac{x+i}{x^2+1} dx = \left[\frac{1}{2}\ln(x^2+1) + i\tan^{-1}x\right]_{-1}^{1} = i[\tan^{-1}1 - \tan^{-1}(-1)] = \frac{i\pi}{2}$$

BC: $\quad w = \dfrac{1}{1 + iy} \;=\; \dfrac{1 - iy}{1 +\; y^2}$

$$\int_{\mathrm{B}}^{\mathrm{C}} w dz = i\int_{-1}^{1} \frac{1-iy}{1+\;y^2} dx = i\left[\tan^{-1}y \;-\; \frac{1}{2}\ln(1+\;y^2)\right]_{-1}^{1} = \frac{i\pi}{2}$$

CD: $\quad w = \dfrac{1}{x + i} \;=\; \dfrac{x - i}{x^2 \;+\; 1}$

$$\int_{\mathrm{C}}^{\mathrm{D}} w dz = \int_{1}^{-1} \frac{x-i}{x^2+1} dx = \left[\frac{1}{2}\ln(x^2+1) - i\tan^{-1}x\right]_{1}^{-1} = \frac{i\pi}{2}$$

DA: $w = \dfrac{1}{-1+iy} = -\dfrac{1+iy}{1+y^2}$

$$\int_D^A wdz = -i\int_1^{-1} \frac{1+iy}{1+y^2}dx = -i\left[\tan^{-1} y + \frac{1}{2}\ln(1+y^2)\right]_1^{-1} = \frac{i\pi}{2}$$

And so the contour integral of z^{-1} counterclockwise around the whole square is $2\pi i$. It is not zero, because z^{-1} is not analytic – is has an infinity at the origin of coordinates.

We get the same result if we integrate counterclockwise around the unit circle

$z = e^{i\theta}$, $dz = ie^{i\theta}d\theta$, $w = e^{-i\theta}$.

Then:

$\oint wdz = \int_0^{2\pi} e^{-i\theta} . ie^{i\theta}\, d\theta = 2\pi i.$

At this point, we understand that the contour integration around a closed circuit of a function that is analytic everywhere around the perimeter of the closed circuit, as well as analytic everywhere inside the closed circuit, is zero.

We also understand that the contour integral around a closed circuit of a function that is not everywhere analytic is not zero, and its sign depends upon whether we integrate clockwise or counterclockwise around the closed circuit. One way in which a function may not be everywhere analytic on or within the circuit is that the function may have one or more singularities (infinities) inside the circuit. Another way in which a function might be nonanalytic would be if the function were not single-valued.

Some time in the dim and distant future, if and when I have the time and the inclination (watch this space!) I may tell you that, in the former case, the contour integration aroud the closed circuit is equal to <u>$2\pi i$ times the sum of the residuals at each of the singularities</u>. But then, of course, I shall then have to explain what are meant by the residuals – and how to calculate them. In the yet further dim and distant future, we may discuss how to integrate a function that is not single-valued. Sorry to leave it there – but I'll try and get round to it one of these days.