Chapter 13
Complex Numbers

1. *Introduction*

In this site, devoted to integration, I thought I should include the topic of integration of functions of a complex variable. Before tackling how to integrate a function of a complex variable, it would be well to understand what a function of a complex variable is. And before tackling that, it is well to know what a complex variable, or a complex number, is. Thus, before getting down to how to integrate such a function, we devote a chapter (this one) to a review of complex numbers, followed by a chapter on functions of a complex variable, and then a chapter on differentiating a function of a complex variable, before getting down to our main business of integration of a function of a complex variable.

Many readers of these notes will already be familiar with complex numbers and variables, so this chapter will be short and sweet, enough to help readers to become familiar with the notation I use and with the way I handle complex numbers and variables.

2. *Algebra of number pairs*

In the ordinary algebra with which we are all familiar, the elements are ordinary ("real") numbers such as 3, 0, −1.5, 1000000, $\sqrt{2}$, π, *e*, or variables, such as *x, y* or constants such as *a*, *b*, together with operators suich as =, +, –, ×, ÷, to which we give their precise, familiar meanings.

Now let us invent, purely for the fun of it, a new algebra, in which the elements, instead of being single numbers, are pairs of numbers, such as (3,7), (1,0), (2, −7), (*e* , π). We'll call such number pairs *complex numbers*. We'll invent some operations, such as =, +, –, ×, ÷. Let's define = to mean

$$(a\,,b) \;=\; (c\,,d) \quad \text{iff} \;\; a = c \;\; \text{and}\; b = \; d \qquad 13.2.1$$

Here "iff" means "if and only if". This meaning of = looks like a very obvious and almost trivial meaning to give to the symbol; but it is a very important meaning and should never be forgotten.

Let us define + and – so that

$$(a,b) \;+\; (c\,,d) \;=\; (a+c\,,\; b+d) \qquad 13.2.2$$

$$(a,b) \;-\; (c\,,d) \;=\; (a-c\,,\; b-d) \qquad 13.2.3$$

Now we could elect to define × and ÷ in a similar and obvious way, but it turns out that that doesn't lead to a very interesting or productive sort of algebra. Instead, we are going to define the symbol × to mean

$$(a,b) \times (c,d) = (ac - bd, bc + ad) \qquad 13.2.4$$

This is not something that needs to be proved; we just decide that that is what the symbol + is to mean. A little later we'll see that it leads to a very useful algebra. It is not an equation to be memorized, either. We'll probably never see it again after this page.

As for the meaning of ÷, well, I've never divided anything by a complex number in my life, and I'm too old to start now. But, if you want to define ÷ as being the opposite of ×, in the sense that if $(a,b) \div (c,d) = (u, v)$ then $(a,b) = (c,d) \times (u, v)$, then the symbol ÷ has to be defined so that

$$(a,b) \div (c,d) = \left(\frac{ac + bd}{c^2 + d^2}, \frac{bc - ad}{c^2 + d^2}\right) \qquad 13.2.5$$

I have never had to use this, let alone memorize it.

Let us try a few examples:

$(3,4) \times (5,6) = (-9,38)$
$(x,0) + (y,0) = (x + y, 0)$
$(x,0) \times (y,0) = (xy, 0)$
$(x,0) \div (y,0) = (x/y, 0)$

We can compose many more examples. We soon find that, as far as the operations =, +, –, ×, ÷, go, number pairs of the form $(a, 0)$ behave exactly like the real numbers that we are familiar with. Indeed if we wish, we could use use the symbol *a* merely as an abbreviation for $(a, 0)$.

Try a few more examples:

$(0,x) + (0,y) = (0,x + y)$
$(0,x) \times (0,y) = (-xy,0)$
$(0,1) \times (0,1) = (-1,0)$

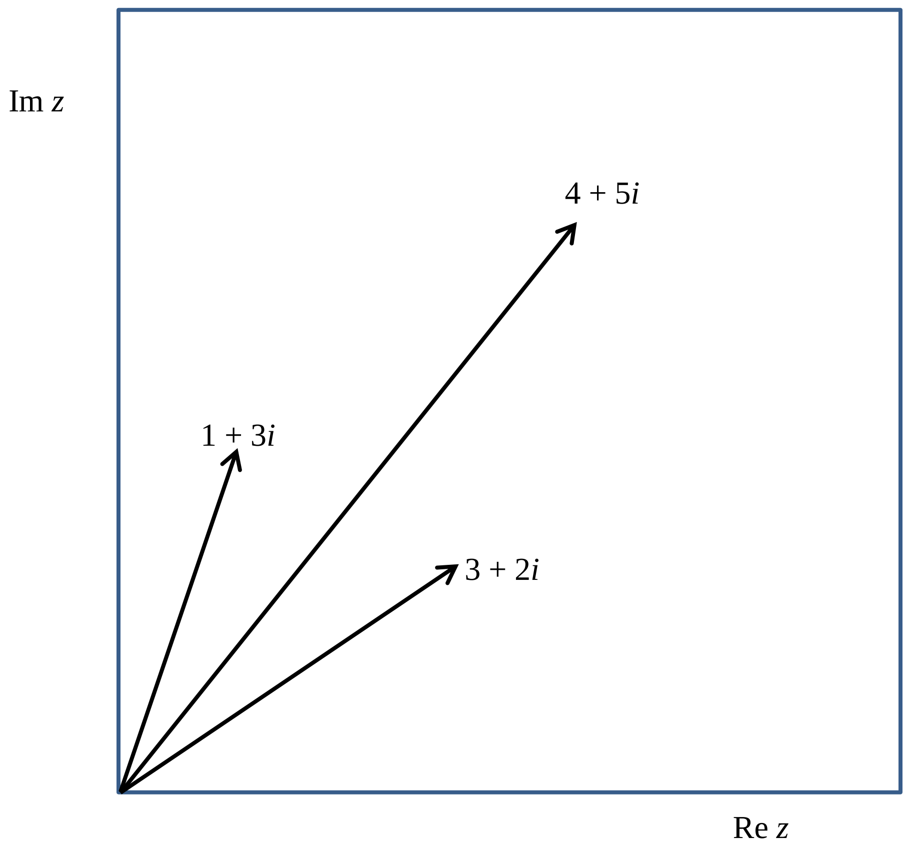
The complex number (0,1) is written for short as *i*, so that the last equation, in shorthand, is written as $i^2 = -1$. Thus *i* is sometimes called "the square root of minus one". Whether *i* really is the square root of minus one, or whether minus one really has a square root, I leave for lunchtime debate.

The numbers $(x, 0)$ and $(0, y)$ can be written for short *x* and *iy* respectively. Their sum is

$(x, 0) + (0, y) = (x, y) = x + iy$. From this point on, a complex number will always be written in the form $x + iy$. The symbol *z* may often be used for $x + iy$. Complex numbers conform to all the ordinary rules of algebra, together with $i^2 = -1$. The first part of a complex number is called the *real* part; the second is the *imaginary* part. Thus Re $z = x$ and Im $z = y$. Whether one part is actually more or less real or more or less imaginary than the other, I leave to the reader to debate with him/herself.   You might also ask yourself whether all mathematical equations and formulas are merely products of the human imagination, or whether they already exist in the Universe independently of human thought. Are mathematical equations invented or discovered? I'm afraid I don't know the answer to such weighty quations.

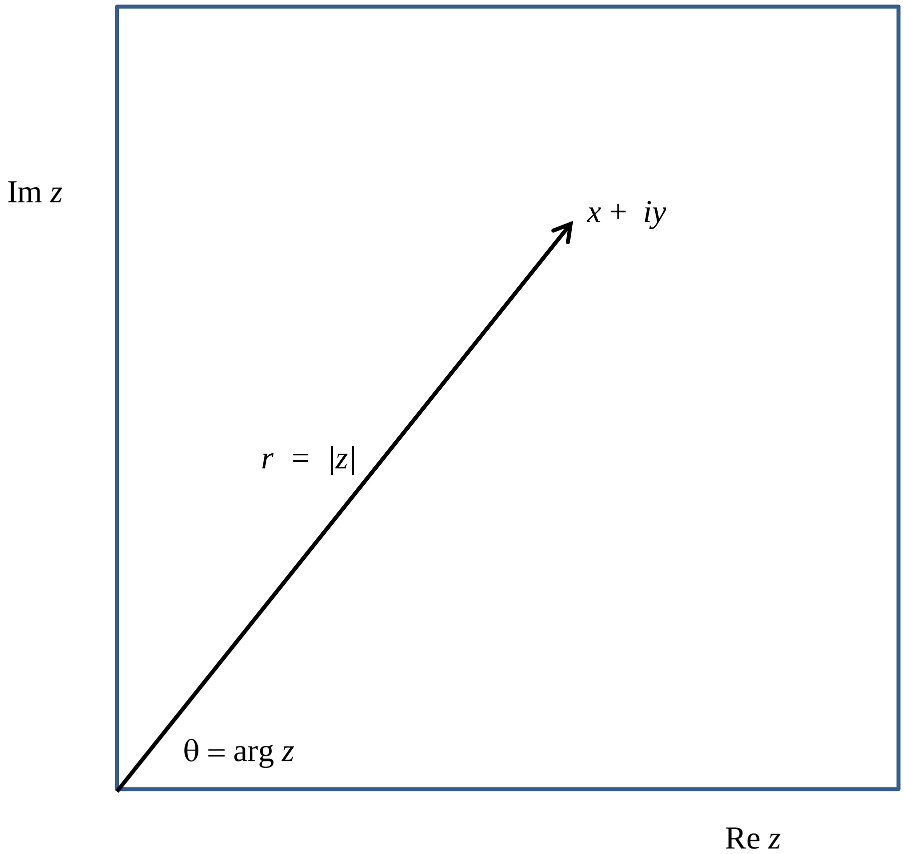
3. *The Argand Diagram*

Complex numbers *add* in a similar manner to the addition of vectors in 2-space. However, complex numbers are not vectors and they do not multiply as vectors do. Since they do, however, *add* as vectors, we can, for many applications, represent a complex number by a vector in 2-space. Such a drawing in the complex plane, is called an *Argand diagram*. We show below the summation of the complex numbers $3 + 2i$ and $1 + 3i$ to form the number $4 + 5i$.



4. *Polar Coordinates*

We can describe a complex number by specifying its real and imaginary components. Alternatively, we can use polar coordinates, and descibe a complex number by specifying its *modulus* $r = |z|$ and its *argument* $\arg z = \theta$.



Thus a complex number $z = x + iy$ can also be written

$$z = r(\cos\theta + i\sin\theta) \qquad 13.4.1$$

where the modulus $r = |z| = \sqrt{x^2 + y^2}$, 13.4.2

and the argument is $\theta \quad \tan^{-1}(y/x)$. 13.4.3

It is often convenient to write for short

$$\cos\theta + i\sin\theta = \text{cis}\,\theta \ . \qquad 13.4.4$$

Thus a complex number can be written as r cis θ.

This notation (cis) may not be universally understood, so it may be well, if you use it, to define it on first usage.

The *conjugate* of a complex number $z = x + iy$ is $z^* = x - iy$. It is easy to see that

$$zz^* = x^2 + y^2 \qquad 13.4.5$$

5. *Division by a complex number or variable.*

I have never done such a thing, though I suppose one could use equation 13.2.3 if you really wanted to.

Suppose that you are deep into some weighty problem in theoretical physics and you have arrived at a hideously complicated expression such as

$$\frac{\Delta\exists\nexists\in\ni\ \delta\aleph\beth\sqrt[3]{\partial}\complement\uparrow\int\ddot{a}}{\cong\leq\ \forall\tau(a-ib)\sum\{\sqrt{u}\vartheta}$$

You notice that there is a complex expression in the denominator, and you have no idea what to do about it. No matter what the context, you don't have to think for one moment what your next move is: Immediately multiply top and bottom by the conjugate of the complex expression:

$$\frac{\Delta\exists\nexists\in\ni\ \delta\aleph\beth\sqrt[3]{\partial}\complement\uparrow\int\ddot{a}\,(a+ib)}{\cong\leq\ \forall\tau(a^2+\ b^2)\sum\{\sqrt{u}\vartheta}$$

You no longer have a complex expression in the denominator and you no longer have to worry about what to do about it. You can get on with the physics now.

Now that you have understood that, look at this:

$$\frac{1}{2i}$$

I hope you didn't look at it for more than a second before you multiplied top and bottom by $-i$.

6. *Maclaurin expansions*

You may remember the expansions

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots \qquad 13.6.1$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \qquad 13.6.2$$

$$e^{\theta} = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots \qquad 13.6.3$$

From these, you will be able to discover or to verify that

$$\cos\theta + i\sin\theta = e^{i\theta} \qquad 13.6.4$$

and by adding and subtracting cis θ and its conjugate it is a short step to show that

$$\cos\theta = \tfrac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \qquad 13.6.5$$

and

$$\sin\theta = -\tfrac{i}{2}\left(e^{i\theta} - e^{-i\theta}\right) \qquad 13.6.6$$

7. *Hyperbolic functions*

We do not attempt a huge comprehensive treatise on hyperbolic functions here. Rather, we give a brief reminder of the definitions of sinh and cosh and their relations with sin and cos, enough to follow what will come in a later section.

Those who are familiar with hyperbolic functions may recall that

$$\cosh\theta = \tfrac{1}{2}\left(e^{\theta} + e^{-\theta}\right) \qquad 13.7.1$$

so that $$\cosh i\theta = \cos\theta\,, \qquad 13.7.2$$

and $$\sinh\theta = \tfrac{1}{2}\left(e^{\theta} - e^{-\theta}\right) \qquad 13.7.3$$

so that $$\sinh i\theta = i\sin\theta \qquad 13.7.4$$

Inverse hyperbolic functions.

If $$x = \tfrac{1}{2}(e^{y} + e^{-y}) = \cosh y \qquad 13.7.4$$

then $$y = \cosh^{-1}x \qquad 13.7.5$$

All we have to do is to invert equation 13.7.4 and express *y* as a function of *x*.
Let $p = e^{y}$, then equation 13.7.4 is a quadratic equation in *p*: $p^2 - 2xp + 1 = 0$, with solutions $p = x \mp \sqrt{x^2 - 1}$.

Thus $$y = \cosh^{-1}x = \ln\left(x \mp \sqrt{x^2 - 1}\right), \qquad 13.7.6$$

which, believe it or not (think about it!), is the same as

$$\cosh^{-1}x = \pm\ln\left(x + \sqrt{x^2 - 1}\right) \qquad 13.7.7$$

This has no real values for $x < 1$, It is two-valued for $x > 1$. It is zero for $x = 1$. Similar considerations, starting from $\sinh x = \frac{1}{2}(e^x - e^{-x})$, will yield

$$\sinh^{-1}x = \ln\left(x + \sqrt{x^2 + 1}\right), \qquad 13.7.8$$

which is single-valued for all real x.

8. *Trigonometry*

You may remember trigonometry as a subject wherein you had to memorize dozens upon dozens of trigonometric identities. One of them, I think, was

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \qquad 13.8.1$$

Do you still remember how to derive it? Have a go! If you still haven't done it after ten minutes, come back here.

Ten minutes have elapsed.

I am now going to teach a course in Trigonometry:

$$\text{cis}\,\theta = e^{i\theta} \qquad 13.8.2$$

That is the end of the course in trigonometry. That is the only equation you need know.

Example

$$\text{cis}\,(A + B) = e^{i(A+B)} = e^{iA}e^{iB} = (\cos A + i\sin A)(\cos B + i\sin B)$$

$$= \cos A \cos B - \sin A \sin B + i(\sin A \cos B + \cos A \sin B)$$

$$= \cos(A + B) + i\sin(A + B)$$

Hence $\quad \cos(A + B) = \cos A \cos B - \sin A \sin B$

and $\quad \sin(A + B) = \sin A \cos B + \cos A \sin B$

Another example

$$\text{cis}\,3A = e^{i3A} = e^{3iA} = \text{cis}^3 A$$

That is: $\cos 3A + i\sin 3A = (\cos A + i\sin A)^3$

$$= \cos^3 A - 3\cos A\sin^2 A + i(3\cos^2 A\sin A - \sin^3 A)$$

Hence $\quad \cos 3A = \cos^3 A - 3\cos A\sin^2 A$

and $\quad \sin 3A = 3\cos^2 A\sin A - \sin^3 A$

We can use the theorem of Pythagoras ($\sin^2 A + \cos^2 A = 1$) to write these as

$$\cos 3A = \cos A(4\cos^2 A - 3)$$

$$\sin 3A = \sin A(3 - 4\sin^2 A)$$

We can quickly express cos 4*A*, cos 5*A*, sin 4*A*, sin 5*A*, etc., in terms of powers of cos *A*, sin *A*, in a similar fashion.

A third example

Can we go the other way, and express powers of sin *A* and cos *A* in terms of sines and cosines of multiples of *A?*

Let $z = \cos A + i\sin A$, so that $r = |z| = 1$

In that case $1/z = z^* = \cos A - i\sin A$, $\quad z + 1/z = 2\cos A$, $\quad z - 1/z = 2i\sin A$.

Then $(z + 1/z)^3 = z^3 + 1/z^3 + 3(z + 1/z)$

and so $\quad 8\cos^3 A = 2\cos 3A + 6\cos A$
or $\quad 4\cos^3 A = \cos 3A + 3\cos A$

In a similar manner, starting from $(z - 1/z)^3$, we quickly arrive at

$$4\sin^3 A = 3\sin A - \sin 3A$$

9. $e^{i\pi}$, *etc.*

Draw the position of $e^{i\pi}$ on the Argand diagram. (What is its modulus? What is its argument?) Alternatively $e^{i\pi} = \text{cis}\,\pi$, so what are its real and imaginary parts? Either way, you will find this interesting relation between the three most interesting numbers in mathematics: *e*, *i*, and π:

$$e^{i\pi} = -1 \qquad 13.9.1$$

You'll also find that $$e^{i\pi/2} = i \qquad 13.9.2$$

and that $$i^i \cong 0.2079 \qquad 13.9.3$$

10. *Logarithms*

I use the symbol ln for the natural logarithm – i.e. $\log_e$

Examples: ln 2 = 0.6931 ln 1 = 0 ln 0.5 = –0.6931

But what is ln −2 ?

Answer: $\ln(-2) = \ln[2 \times (-1)] = \ln 2 + \ln -1 = \ln 2 + \ln e^{i\pi} = \ln 2 + i\pi$

That is: $$\ln(-2) = 0.6931 + 3.1416i$$

And if a is any positive real number, $\ln -a = \ln a + i\pi$. That is, unless $a = 1$, $\ln -a$ is complex. If $i = 1$, ln −1 is wholly imaginary. By that, I don't mean that it is just a product of my imagination, that it is all my mind. I mean that it has no real component; or the real part of it (thought of as complex number) is zero. Thus $\ln -1 = i\pi$. Wholly imaginary.

So we know the natural logarithm of a positive real number, and of a negative real number. Also, if $a \to 0$, $\ln a \to -\infty$.

What is the logarithm of an imaginary number?

$\ln 2i = \ln 2 + \ln i = \ln 2 + \ln e^{i\pi/2} = \ln 2 + i\pi/2.$
Like ln −1, ln i is wholly imaginary: $i\pi/2$

But what is ln z, where $z = x + iy$, a complex number? We have to wait until Chapter 14, Functions of a Complex Variable.

11. *More trigonometry*.

In Section 9, we agreed that an expression such as ln (–2) has no meaning within the familiar algebra of real numbers and variables. However, in our algebra of number pairs (which we now call complex numbers), there is no difficulty. We found that ln (–2) is merely ln 2 + $i\pi/2$.

How about something such as $\cos^{-1}2$ or $\sin^{-1}2$? That has no meaning within the algebra of real numbers and variables. I wonder if it has some meaning in the world of complex numbers. Let's find out.

Solve $\cos x = 2$.
Solution: $\cosh ix = 2$
$$ix = \cosh^{-1}2 = \pm\ln\left(2 + \sqrt{3}\right) = \pm 1.317$$
$$x = \mp 1.317i$$

Solve $\sin x = 2$.
Solution: $\cos(\pi/2 - x) = 2$
$\pi/2 - x = \cos^{-1}2$
$x = 1.571 \pm 1.317\, i$