CHAPTER 6
General Quadratic Equation, Part II
$\Delta_3 = 0$: Noncentral Surfaces
Planes

6.1 *Introduction*

We start with the general quadratic equation in three variables:

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0. \qquad 6.1.1$$

In this chapter we suppose that $\Delta_3 = 0$. We recall from Chapter 5 (which also defines what we mean by Δ_3) that $\Delta_3 = 0$ implies that the equation does not represent a central quadric surface such as an ellipsoid or hyperboloid, but it represents a surface that does not have a centre of symmetry, such as two planes, or a paraboloid. If $\Delta_3 = 0$, we cannot, by a translation of axes, find a set of coordinate axes such that the coefficients u, v, w are zero. In this chapter, we shall deal with the case where equation 6.1.1 represents two planes.

Other possibilities, in addition to a paraboloid or a pair of planes, is that the equation might represent a cylinder (elliptic or hyperbolic), or perhaps a single straight line or perhaps no point can satisfy the equation. These cases will be discussed in Chapter 8.

6.2 *Two Planes*

[Before reading this section, it will be good to remind yourself of equation 2.2.4. Also, have a good program handy so that you instantly solve a quadric equation in one variable.]

When I was at school, the teacher used to make us factorize quadratic expressions such as $15x^2 - 11xy - 14y^2$ into two linear factors. I always found such problems difficult [an answer is $(3x + 2y)(5x - 7y)$], and I don't find them much easier today. The expression $15x^2 + 11xy + 14y^2$ cannot be split into two real linear terms, and indeed one of the problems, before trying to split such an expression, is to determine before starting whether it can be split at all.

Example 1

If we are given an expression like equation 6.1.1, such as, for example,

$$12x^2 - 20y^2 - 21z^2 + 47yz - 36zx + 14xy - 22x + 7y + 5z + 6 = 0, \qquad 6.2.1$$

for which $\Delta_3 = \begin{vmatrix} 12 & 7 & -18 \\ 7 & -20 & 23.5 \\ -18 & 23.5 & -21 \end{vmatrix} = 0$,

can it be split into two real linear factors (in which case the equation represents two planes), and, if so, how do we do it? This is how I did it - maybe there's an easier way. If anyone can suggest a better method, please let me know [jtatum at uvic dot ca]. In the meantime, don't read on unless you are sitting in front of your computer with your program for solving a quadratic equation ready for instant use.

By successively putting $y = z = 0, \quad z = x = 0, \quad x = y = 0$, I found that the planes contain the following six points on the coordinate axes:

x - axis	y - axis	z - axis
A_2: $(+\frac{3}{2}, 0, 0)$	B_2: $(0, +\frac{3}{4}, 0)$	C_2: $(0, 0, -\frac{3}{7})$
A_1: $(+\frac{1}{3}, 0, 0)$	B_1: $(0, -\frac{2}{5}, 0)$	C_1: $(0, 0, +\frac{2}{3})$

There are four ways in which two planes can contain these six points. I list them below, together with the equations that represent them. (For how, see equation 2.2.4.)

$$\begin{array}{llll} \mathrm{A_1B_1C_1}: & 6x - 5y + 3z - 2 = 0 & \mathrm{A_2B_2C_2}: & 2x + 4y - 7z - 3 = 0 \\ \mathrm{A_1B_2C_1}: & 18x + 8y + 9z - 6 = 0 & \mathrm{A_2B_1C_2}: & 4x - 15y - 14z - 6 = 0 \\ \mathrm{A_1B_1C_2}: & 18x - 15y - 14z - 6 = 0 & \mathrm{A_2B_2C_1}: & 4x + 8y + 9z - 6 = 0 \\ \mathrm{A_1B_2C_2}: & 9x + 4y - 7z - 3 = 0 & \mathrm{A_2B_1C_1}: & 4x - 15y + 9z - 6 = 0 \end{array} \qquad 6.2.2$$

Only the first of these pairs of linear equations, when multiplied together, yield the original quadratic equation, and these are therefore the sought-after factors. The original quadratic equation therefore represents the two planes

$$\begin{array}{l} 6x - 5y + 3z - 2 = 0 \\ 2x + 4y - 7z - 3 = 0 \end{array} \qquad 6.2.3$$

Let us now try four more examples, each of which has $\Delta_3 = 0$, but each has a slightly different wrinkle.

Example 2

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$$9x^2 + 49y^2 + 4z^2 - 28yz + 12zx - 42xy + 42x - 98y + 28z + 45 = 0. \qquad 6.2.4$$

$$\Delta_3 = \begin{vmatrix} 9 & -21 & 6 \\ -21 & 49 & -14 \\ 6 & -14 & 4 \end{vmatrix} = 0\,.$$

By successively putting $y = z = 0$, $z = x = 0$, $x = y = 0$, it is found that the planes contain the following six points on the coordinate axes:

$$\begin{array}{llll} A_1: & (-3, 0, 0) & A_2: & (-\tfrac{5}{3}, 0, 0) \\ B_1: & (0, \tfrac{5}{2}, 0) & B_2: & (0, \tfrac{9}{7}, 0) \\ C_1: & (0, 0, -\tfrac{9}{2}) & C_2: & (0, 0, -\tfrac{5}{2}) \end{array}$$

Again, there are four possible pairs of planes that connect six points. I list only the pair $A_1B_2C_1$ and $A_2B_1C_2$:

$$A_1B_2C_1: \; 3x - 7y + 2z + 9 = 0 \qquad A_2B_1C_2: \; 3x - 7y + 2z + 5 = 0. \qquad 6.2.5$$

These are the two parallel planes that we dealt with in Section 2.5, and which we found to be separated by 6.385 m. The other three pairs of planes that can connect the six points yield linear expressions which, when multiplied, do not yield the original quadratic expression.

Example 3

$$4x^2 + 9y^2 + 25z^2 - 30yz - 20zx + 12xy + 24x + 36y - 60z + 36 = 0. \qquad 6.2.6$$

$$\Delta_3 = \begin{vmatrix} 4 & 6 & -10 \\ 6 & 9 & -15 \\ -10 & -15 & 25 \end{vmatrix} = 0$$

The surface cuts each of the *x*-, *y*, and *z*- axes at a single point,
It cuts the *x*-axis at $x = -3$.
It cuts the *y*-axis at $y = -2$.
It cuts the *z*-axis at $z = +1.2$.

By following the procedures of Example 1, you will eventually find that equation 6.2.6 turns out to be

$$(2x + 3y - 5z + 6)^2 = 0, \qquad 6.2.7$$

which you could describe as two coincident planes, or, if you prefer, just one plane.

Example 4

$$2x^2 - 4zx + 5xy - 12x - 15y + 12z + 18 = 0 \qquad 6.2.8$$

This looks interesting, in that some of the usual terms are zero.

$$\Delta_3 = \begin{vmatrix} 2 & 2.5 & -2 \\ 2.5 & 0 & 0 \\ -2 & 0 & 0 \end{vmatrix} = 0$$

It cuts the x-axis where $2x^2 - 12x - 6 = 0$, i.e. $x =$ –0.464 and +6.464
It cuts the y-axis at only one point, namely y = –0.6
It cuts the z-axis at only one point, namely z = 0.5
This means that, while one of the planes cuts all three axes, the second plane cuts only the x-axis. Thus one of the planes is perpendicular to the x-axis, so that equation 6.2.8 can be factored into an equation of the form

$$(x - a)(Ax + By + Cz + D) \qquad 6.2.9$$

By comparison of equation 6.2.9 with 6.2.8, we soon find that

$$(x - 3)(2x + 5y - 4z - 6) \qquad 6.2.10$$

Example 5

$$xy + 7x - 4y - 28 = 0 \qquad 6.2.11$$

It will not take long to discover that this is

$$(x - 4)(y + 7) = 0 \qquad 6.2.12$$

so that this, too, is an example of two planes. One of these is perpendicular to the x-axis, and intersects it at one point, and the other is perpendicular to the y-axis, and intersects it at one point. Neither of them touches the z-axis.