Chapter 1. Principles of Planetary Photometry

1. Introduction.

The subject of planetary photometry is, in substantial part, a subset of that branch of mathematical physics known as *radiative transfer*, for which the classical and definitive work is that of Chandrasekhar (1960).

Here we present this aspect of the subject in a modern context and although we have adhered as much as possible to the symbols, nomenclature and notation of Chandrasekhar, the following changes and additions have been made.

(i) The quantity called by Chandrasekhar *intensity* I is here called *radiance* L. I make no apology for this since it conforms with modern international radiometric standards.

(ii) A plane parallel beam of radiation is specified by its *radiant flux density* F rather than its *net flux* $p F$, the latter being a more generally defined quantity.

(iii) Shorthands for incident, reflected and transmitted radiation, with subscripts i, r and t have been introduced.

(iv) Reflectance functions in addition to Chandrasekhar's formulations are presented.

2. Radiance and the Equation of Transfer.

Radiance may be regarded as the fundamental quantity of radiative transfer.

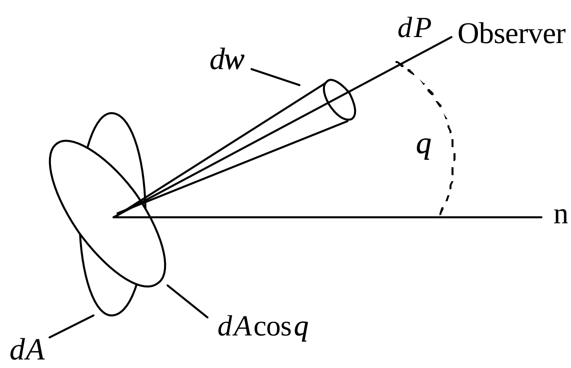


Fig.1.

Consider, as shown in figure 1, an emitting (or reflecting) surface of area dA which emits dP watts of power into solid angle dw about the direction of an observer at angle q to the surface normal vector **n** of dA, the latter presenting a *projected area* $dA\cos q$ to the observer. The radiance detected by the observer is then

$$L = \frac{dP}{dw\, dA \cos q} \qquad (1)$$

Radiance then has the following properties:

> It is defined at a point and in a specified direction.
>
> It is independent of the distance from which it is observed
>
> Its SI units are watts per square metre per steradian ($\mathrm{W\ m^{-2}\ sr^{-1}}$). This can be interpreted in either of two ways. Either, it is the power projected into unit solid angle from unit projected area of an extended surface (i.e. projected on a plane at right angles to the line of sight from the observer); or it is the power arriving per unit area at the observer from unit solid angle (subtended at the observer) of the extended source. That these are equivalent is shown in reference (2).

If radiance is the fundamental quantity of radiative transfer, then the fundamental law is the *equation of transfer*

$$-\frac{dL}{kr\, ds} = L - \Im\ . \qquad (2)$$

Here $\frac{dL}{ds}$ is the rate of change of radiance with, and in the direction of, position ***s*** in a given medium, r is the density of the medium ($\mathrm{kg\ m^{-3}}$) and k is the *mass attenuation coefficient* ($\mathrm{m^{2}\ kg^{-1}}$). Here attenuation refers to any process which reduces the brightness of a beam of radiation, and so includes absorption and scattering. Some authors use the word *extinction* for attenuation, and some (particularly in the field of stellar atmospheres) use the word *opacity* to refer to the mass attenuation coefficient.

Equation (2) could be read as follows: as a beam of radiance L traverses the distance ds it will be diminished in radiance by the amount $krL\,ds$ and enhanced by the amount $kr\Im\, ds$. The quantity $\Im$ is called the *source function* and, as we shall see, a typical problem of planetary photometry is to find a solution for this quantity before solving the equation of transfer as a whole.

3. *Diffuse Reflection and Transmission*.

The fundamental problem of planetary photometry is the diffuse reflection and transmission of a *plane parallel beam* of radiation by a *scattering medium*, which we would understand as a planetary atmosphere and/or surface or a planetary layer such as the rings of Saturn. Such media may be idealised as locally plane parallel strata in which physical properties are uniform throughout a given layer. In such cases we may use a hybrid Cartesian and spherical frame of reference in which the Oxy plane is the surface and z-axis points in the direction of the surface normal. Directions are then specified by the polar and azimuthal angles J and j ("curly theta" and "curly phi") respectively. Further, with problems of this kind, rather than working in actual physical distances it is preferable to work in terms of *normal optical thickness* t, measured downwards from $z = 0$, such that $dt = -kr\,dz$. Radiation that has traversed a path of optical thickness t is attenuated by a factor of e^{-t}.

Using the direction cosine $m = \cos J$ the *standard form* of the equation of transfer for *plane parallel media* is

$$m\frac{dL(t,m,j)}{dt} = L(t,m,j) - \Im(t,m,j) \tag{3}$$

For a scattering medium, the only contribution to the source function is the scattering of that radiation which has been incident on the medium from *external* sources, so that, by totalling the contributions impinging on level t from all directions, the source function is

$$\Im(t,m,j) = \frac{1}{4p}\int_{-1}^{1}\int_{0}^{2p} p(m,j;m',j')L(t,m',j')dj'dm' \tag{4}$$

where p is the *normalised phase function* which determines the angular distribution of the scattering. A convenient way to think of p is that $\frac{p}{4p}dw$ is the probability that a photon travelling in the direction (m',j') would be scattered into an elemental solid angle dw in the direction (m,j).

Radiation traversing a normal optical thickness dt in the direction (m,j) will be attenuated by the amount $dL = Ldt/m$. Of this amount, a fraction can be attributed to that caused by scattering alone – this fraction is called the *single scattering albedo* v_0. It then follows that the phase function p must be normalised according to

$$\int_{4p}\frac{p}{4p}dw = 0 \le v_0 \le 1 \tag{5}$$

and if p is a constant, then $p = v_0$ and the scattering is *isotropic*.

4. Directions and Notation.

The strength of a plane parallel beam of radiation is specified by the radiant flux density F watts per square metre such that F = dP/dA, where A is the area perpendicular to the direction of propagation. Thus F is equal to the net flux pF used by Chandrasekhar, with the important exception that F is used *only* for a plane parallel beam.

Since the equation of transfer deals only in radiances, we will now address the rather intriguing question, “What is the radiance of a plane parallel beam?”

Figure 2 shows a ray of a plane parallel beam of flux density F incident on the surface of a scattering medium. We shall let the resulting incident radiance be L_i, which, using Chandrasekhar’s notation would be specified in position and direction as

$$L_i = L(0,-m_0, j_0), \quad m_0 = |\cos J_0| \tag{6}$$

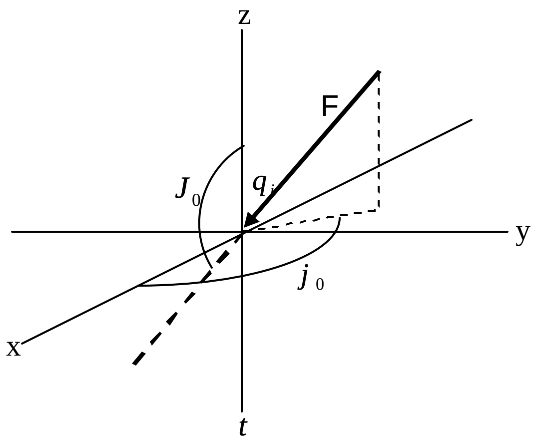


Fig. 2.

Many authors specify the polar direction of this radiation in terms of an *angle of incidence*, say i or q_i, as the angle between the surface normal and the incident ray, such that $i = p - J_0$ and define m_0 as $m_0 = \cos i$.

The angular distribution of F (or, rather, its lack of it) may be specified analytically by making use of the Dirac delta function, which has the property that

$$\int_{-\infty}^{\infty} f(x) d(x-a) dx = f(a)$$

and, perhaps more importantly, for any $e > 0$

$$\int_{a-e}^{a+e} f(x) d(x-a) dx = f(a)\,.$$

The incident radiance on the surface in the direction $(-m_0, j_0)$ is then[1]

$$L_i = \mathrm{F} d(m - m_0) d(j - j_0)\,. \tag{7}$$

Figure 3 shows a reflected beam of radiance L_r where

$$L_r = L(0,+m, j), \quad m = \cos J \tag{8}$$

[1] If equation (7) bothers you in that its right hand side does not appear to have units of radiance, then do not worry; in chapter 2 we will demonstrate that indeed it does have such units.

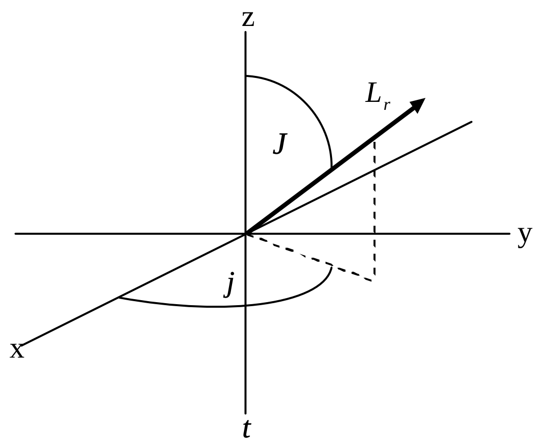


Fig. 3.

Again, many authors define the polar direction of reflection in terms of an *angle of reflection*, the angle between the surface normal and the reflected ray, say q_r (which is the same as J), and define m as cos $q_{r.}$

Figure 4 shows a transmitted ray of radiance L_t where

$$L_t = L(t, -m, j), \quad m = |\cos J| \tag{9}$$

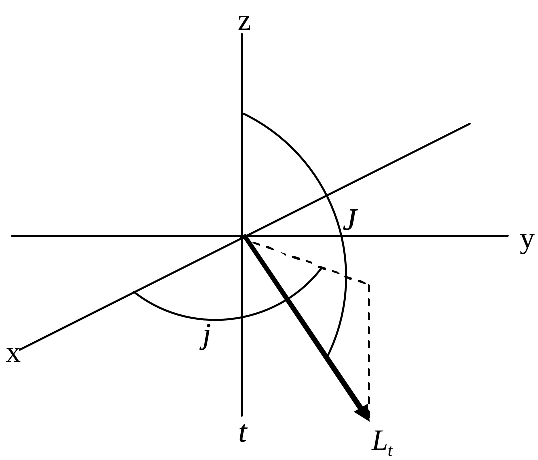


Fig. 4.

The question then arises: are the values of m_0 and m, introduced by Chandrasekhar and taken up by others, albeit with different definitions, always positive? The answer to this is, mostly, yes, but care needs to be taken depending on the context in which they occur, as, for example, in the following cases.

Consider the problem of determining the phase angle $0 \le a \le p$ between incident and reflected or transmitted directions m_0 and m , where

$$\cos a = m_0 m + \sqrt{(1-m_0^2)(1-m^2)}\ \cos(j - j_0)\,. \qquad (10)$$

For reflection both m and m_0 are positive, but for transmitted rays m must be negative. The phase function p is often expressed in terms of a or $\cos a$.

Or, consider, as in figure 5, the optical path lengths of reflected and transmitted radiation from a depth within a medium of optical thickness t. The total optical path for the reflected ray is $t / m_0 + t / m_r$ and for the transmitted ray $t / m_0 + (t - t)/ m_t$ in which m_0, m_r and m_t are all positive.

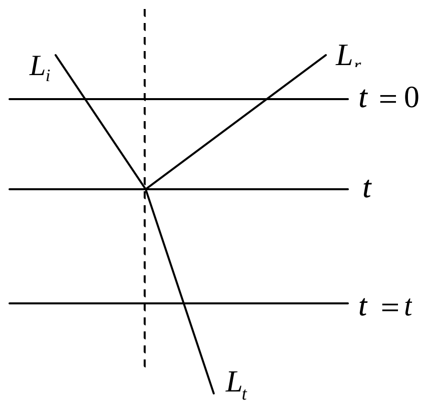


Fig. 5.

5. Reflectance Functions.

In the most general case of diffuse reflection, the reflectance of a surface will depend on both the direction of the incident radiation and that of the reflected radiation. The *bidirectional reflectance distribution function*, f_r, links the *irradiance* E to the reflected radiance, such that

$$L_r = f_r(m, j; m_0, j_0) E(m_0, j_0). \qquad (11)$$

For a surface irradiated with flux density F, the irradiance is simply the component of the flux density perpendicular to the surface

$$E = m_0 F\,, \qquad (12)$$

so that, abbreviated, we can write

$$L_r = f_r m_0 F \qquad (13)$$

One of the simplest examples of a reflectance rule is that of a Lambertian reflecting surface for which the radiance is isotropic, so that

$$L_r = \frac{l_0}{p} m_0 F\,, \qquad (14)$$

where l_0 is sometimes referred to as the *Lambertian albedo*. Although it is not strictly physically correct, it is convenient (Chandrasekhar, p147) to identify l_0 with the single scattering albedo v_0, so for Lambert's law the BRDF is

$$f_r = \frac{v_0}{p}. \qquad (15)$$

For the most part, we shall refer all reflectance rules used to a BRDF; alternative reflectance functions will be discussed in ***§8***.

6. Diffuse Reflection – the Lommel-Seeliger Law.

The Lommel-Seeliger reflectance rule is a time-honoured law which is still very much in use today. It is based on a model which is possibly the simplest from which a solution may be readily obtained for the source function and the equation of transfer. As such it is a *single scattering* model in which the scattering is *isotropic*, *i.e.* $p = v_0$.

Consider a surface irradiated by flux density as shown in figure 3, so that the incident radiance is given by equation (7). Of this incident radiation, only a fraction will penetrate to optical depth t without being scattered or absorbed

$$L(t, m, j) = \mathsf{F} e^{-t/m_0} d(m - m_0) d(j - j_0) \qquad (16)$$

and the source function is

$$\begin{aligned} \Im(t, m, j) &= \frac{1}{4p} \int_{-1}^{1} \int_{0}^{2p} v_0 \mathsf{F} e^{-t/m_0} d(m' - m_0) d(j' - j_0) dj' dm' \\ &= \frac{v_0}{4p} \mathsf{F} e^{-t/m_0} \end{aligned} \qquad (17)$$

Thus the contribution to the radiance from isotropic scattering in the direction $(+m, j)$ from a layer of thickness dt at a depth t is

$$dL = \frac{v_0 \mathsf{F} e^{-t/m_0}}{4pm} dt \qquad (18)$$

so that the radiance emerging from the surface, without incurring any further absorption or scattering, is

$$dL(0, m, j) = \frac{v_0 \mathsf{F} e^{-t/m_0}}{4pm} e^{-t/m} dt \qquad (19)$$

Note that dL is the contribution to the total radiance from the layer resulting from *single scattering*. The Lommel-Seeliger model considers only the scattering of the collimated incident light. It does not take into account scattering of diffuse light which has made its way indirectly to the same position by being scattered one or more times, *i.e.* it does not consider *multiple scattering*.

For a planetary surface, the layer is "semi-infinite" ($\tau = \infty$) and the total radiance in the direction μ is

$$L_r = \frac{\varpi_0 \Phi}{4\pi\mu} \times \int_0^\infty \exp[-\tau(\frac{1}{\mu_0} + \frac{1}{\mu})] d\tau \quad . \tag{20}$$

resulting in

$$L_r = \frac{\varpi_0 \Phi}{4\pi\mu} \frac{\mu_0 \mu}{\mu + \mu_0} \tag{21}$$

and since the irradiance is $E = \Phi\mu_0$ and $L_r = f_r E$ it follows that the bidirectional reflectance distribution function (BRDF) which defines the Lommel-Seeliger reflectance rule is

$$f_r = \frac{\varpi_0}{4\pi} \frac{1}{\mu + \mu_0} \; . \tag{22}$$

It is interesting to note that Chandrasekhar never *quite* derives the Lommel-Seeliger law formally and explicitly; indeed the name is not even mentioned. However, he does come very close on at least two occasions – see Chandrasekhar p146 and p217.

7. Other Reflectance Functions.

It is important to distinguish between a reflectance function and the reflectance law it represents. So far we have only considered one such function, the BRDF, so that the Lommel-Seeliger law expressed in terms of the BRDF is given by equation (23) and the specific equation for the radiance is given by

$$L_r = \frac{\varpi_0}{4\pi} \frac{1}{\mu_0 + \mu} \mu_0 \Phi \tag{23}$$

Chandrasekhar takes a quite different approach, linking the radiance to the incident flux density through a factor $1/4\mu$, providing a consistent set of *scattering functions* *S* and *transmission functions T*, so that in the case of reflection from a semi-infinite surface we have

$$L_r = \frac{F}{4\mu} S(\mu, \varphi; \mu_0, \varphi_0) = \frac{\Phi}{4\pi\mu} S(\mu, \varphi; \mu_0, \varphi_0) \, , \tag{24}$$

where Chandrasekhar *always* uses $p F$ for incident radiant flux density F. Although, at least at first sight, this formulation may seem strange, even counterintuitive, there is a reason for it; the m in the denominator is used to satisfy the *Helmholtz principle of reciprocity* (Chandrasekhar, p171), so that

$$S(m, j; m_0, j_0) = S(m_0, j_0; m, j). \tag{25}$$

Comparing equations (23) and (24), it follows that for the Lommel-Seeliger law the Chandrasekhar scattering function is

$$S(m, m_0) = \frac{v_0 m_0 m}{m_0 + m}, \tag{26}$$

where it can be seen that the reciprocity principle does indeed hold.

Another function to be found in the literature is the *bidirectional reflectance r*, which links the radiance to the incident flux density, so that the Lommel-Seeliger law is then

$$r(m_0, m) = \frac{v_0}{4p} \frac{m_0}{m_0 + m}, \quad L_r = r\mathsf{F} \tag{27}$$

So, which, if any, of the above functions is the "best" for planetary applications? There does not appear to be any "standard" in use in the literature, indeed the situation would seem to be quite the opposite, many authors making up their own *ad hoc* "reflectance" or "scattering" functions to suit the problem at hand. (This can make for very frustrating reading, especially when words such as "flux", "intensity" and "brightness" are used loosely, as, sadly, is often the case).

The author can see no compelling reason to prefer one function over another. What is important is for authors to state clearly and without ambiguity the properties of the reflectance function and rule(s) which they are using.

8. Diffuse Reflection and Transmission.

A scattering layer of finite optical thickness t may be used to model *e.g.* a planetary ring. If we use the Lommel-Seeliger model, then the reflected radiance of such a layer may be determined by changing the upper limit of the integral in equation (20) so that

$$L_r = \frac{v_0 \mathsf{F}}{4pm} \times \int_0^t \exp[-t(\frac{1}{m_0} + \frac{1}{m})]dt \tag{28}$$

resulting in

$$L_r = \frac{v_0}{4p}\frac{1}{m + m_0} \times$$
$$[1 - \exp\{-t(\frac{1}{m_0} + \frac{1}{m})\}]m_0 \mathsf{F} \tag{29}$$

For the transmitted radiance, it is readily shown that

$$dL_t = \frac{v_0 \mathsf{F} e^{-t/m_0}}{4pm} e^{-(t-t)/m} dt \tag{30}$$

and in the special case $m = m_0$, integration results in

$$L_t = \frac{v_0 \mathsf{F} t}{4pm_0} e^{-t/m_0} \tag{31}$$

and otherwise

$$L_t = \frac{v_0 \mathsf{F}}{4p}\frac{m_0}{m - m_0}[e^{-t/m} - e^{-t/m_0}] \tag{32}$$

In all cases the values of m and m_0 are positive; some authors even explicitly put in absolute value symbols to emphasise this point!

9. Radiances of Planetary Spheres.

We conclude this chapter by applying some of the work covered so far to a planetary situation. We will consider two hypothetical planets idealised as smooth spheres. One planet will have a surface reflecting according to Lambert's law, the other the Lommel-Seeliger law. The observer is able to resolve both planets equally well, so that we may compare and contrast the distribution of radiance across the projected discs at various phase angles.

Consider a unit sphere centred at the origin of an O*xyz* coordinate system irradiated with flux density F from the x-direction. A distant observer in the *xy* plane detects the radiance at phase angle a (the angle Sun-planet-Earth). Using the spherical coordinates $(1, \Theta, \Phi)$ for the surface of the sphere, it can be shown for the *angles of incidence and reflection*

$$\begin{aligned} m_0 &= \sin\Theta\cos\Phi \\ m &= \sin\Theta\cos(\Phi - a) \end{aligned} \tag{33}$$

The projected sphere, the disc seen by the observer, will have projected coordinates

$$\begin{aligned} y' &= \sin\Theta\sin(\Phi - a) \\ z &= \cos\Theta \end{aligned} \tag{34}$$

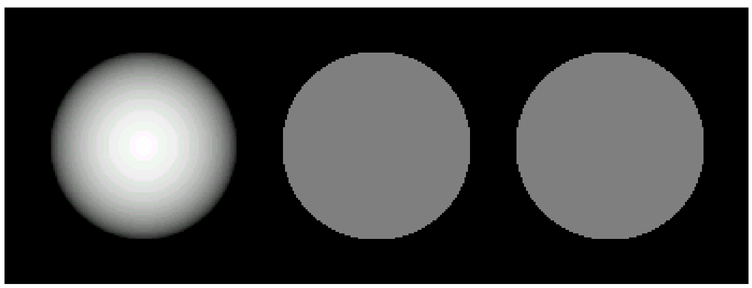
such that $y'^2 + z^2 \le 1$. Except at zero phase, not all the illuminated surface will be visible, since for each point on the disc both the condition $m_0 > 0$ and $m > 0$ must be satisfied in order that the point be both irradiated and not obscured from the observer.

Defining a quantity *relative radiance*, $p\,L / v_0 F$, we can directly compare the radiances of the two spheres, as shown in the table.

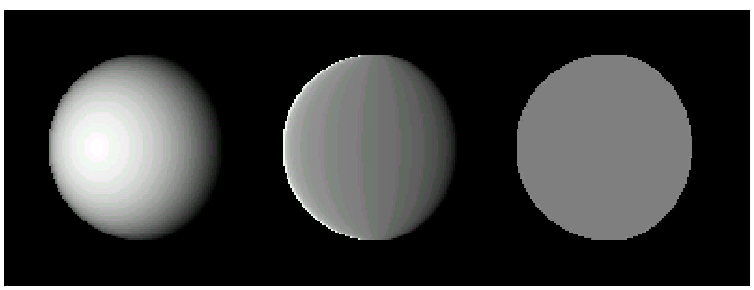
Relative Radiances of Spheres

Lambertian	m_0
Lommel-Seeliger	$\frac{1}{4}\frac{m_0}{m_0+m}$

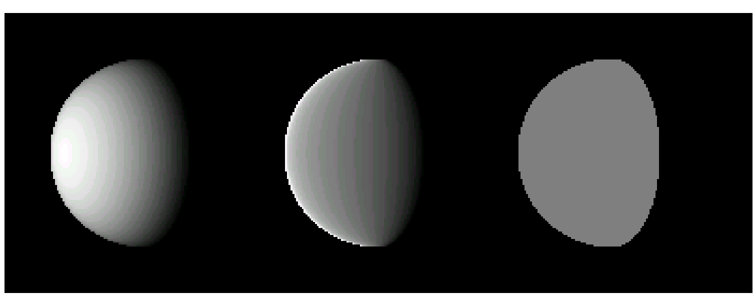
The resulting images are readily programmed. In those that follow (leftmost Lambertian, middle Lommel-Seeliger), each planet has been adjusted so that the maximum relative radiance is *white*. The rightmost image shows the outline of the lune visible to the observer.



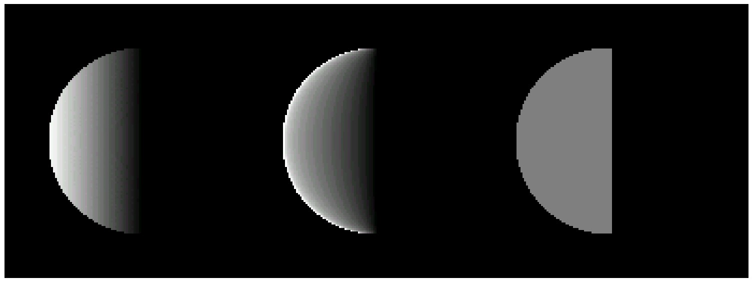
α = 0



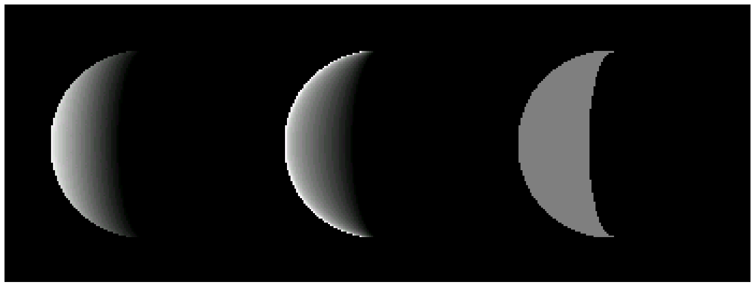
α = 30°



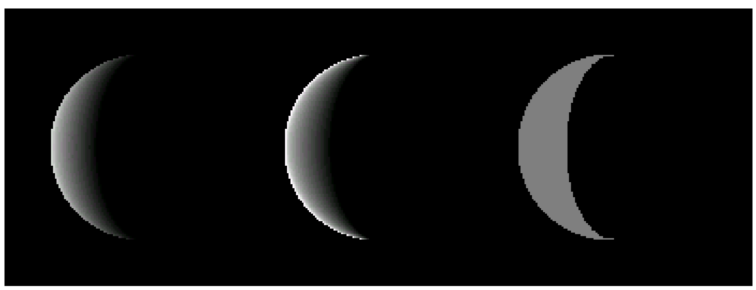
α = 60°



$\alpha = 90^o$



$\alpha = 105^o$



$\alpha = 120^o$

At opposition the Lambertian sphere is limb-darkened, whereas the Lommel-Seeliger sphere is uniformly bright. As the phase angle increases from zero the Lommel-Seeliger sphere becomes darkened towards the terminator and brightened at the limb. For phases greater than ninety degrees, the cusps of the Lommel-Seeliger sphere are more persistant than the Lambertian. Without commenting any further, the images *do* make interesting comparisons with the phases of the Moon.

Reference Notes.

Sections 2, 4, 5, 7, 8 and 9 are based on the author's interpretation of Chandrasekhar's book, chapters I, III, VI and IX.

1. Chandrasekhar, S., 1960, *Radiative Transfer*, Dover, New York.

The ideas of a quantity F defined only for a plane parallel beam and the use of the BRDF (bi-directional reflectance distribution function) for astronomical applications are taken from

2. Lester. P. L., McCall, M. L. & Tatum, J. B., 1979, *J. Roy. Astron. Soc. Can*., **73**, 233

who use F for flux density, which clashes with the F of the pF used by Chandrasekhar – this is the reason for using F. (See also Nicodemus, F.E., *Applied Optics*, **4**, 767 (1965) and **9**, 1474 (1970) – JBT)

Section 10 is a revised and corrected adaptation of an article by the author

3. Fairbairn, M. B., 2002, J. Roy. Astron. Soc. Can., **96**, 18.

Depending on the hardware used, the images shown may display some spurious contouring.