Chapter 6
Simultaneous Differential Equations

What I have in mind here are pairs of equations in two variables (such as x and y, or r and θ) and their derivatives $\dot{x}$ and $\dot{y}$ with respect to some parameter t (which may be the time), in which $\dot{x}$ means $\frac{dx}{dt}$ and $\dot{y}$ means $\frac{dy}{dt}$. An example would be

$$\dot{x} - 3x + y = 0$$
$$\dot{y} - x - y = 0$$

The equations may be considered solved (integrated) when there are no derivatives and the "answers" can be written in a form such as

$$x = f(t)$$
$$y = g(t)$$

In some cases, it may be possible to go further and eliminate t and so obtain a relation between y and x, such as $y = F(x)$ or $G(x, y) = 0$.

In the above two first-order differential equations, there are two derivatives, and so the solution will have two arbitrary constants of integration. The solution, in fact, as we shall find out in due course, can be written as

$$x = (A + Bt)e^{2t}$$
$$y = [A + B(t - 1)]e^{2t}$$

which you can verify (do so!) by substitution in the original equations.

Equations of this sort are not uncommon in physical situations. For example, it is well known (we hope!) that the equations for the radial and transverse components of acceleration in polar coordinates are, respectively:

$$\ddot{r} - r\dot{\theta}^2$$
$$r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

If a particle of small mass m is in orbit around a mass M (e.g., a planet in orbit around the Sun) the force on it towards the Sun will be $\frac{GMm}{r^2}$, while there will be no transverse force. Its equations of motion are, then,

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{GMm}{r^2}$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0.$$

This, then, is an example of the sort of equations to be discussed in this chapter. There are special ways of dealing with these two, involving energy and angular momentum, and the solution is discussed in detail in Chapter 3 of my notes on Celestial Mechanics (orca.phys.uvic.ca/~tatum/celmechs.html). The solution is a conic section, and the two arbitrary constants that arise are determined by the initial energy and angular momentum of the system.

Let us get back, however, to some simpler equations. In particular, let's look at

$$\dot{x} - 3x + y = 0$$

$$\dot{y} - x - y = 0$$

These can be written

$$(D - 3)x + y = 0$$

$$(D - 1)y - x = 0,$$

in which D stand for the operator $\frac{d}{dt}$.

Eliminate y:

$$(D^2 - 4D + 4)x = (D - 2)^2 x = 0$$

We are now on familiar ground, for this is the type of equation discussed in Chapter 4 with $b^2 = 4ac$ and the solution is

$$x = (A + Bt)e^{2t}.$$

There are two ways now to find y. We do the same thing as we did to find x, except that this time we eliminate x from the two original differential equations, and we obtain

$$(D^2 - 4D + 4)y = (D - 2)^2 y = 0,$$

with solution

$$y = (C + Et)e^{2t}.$$

Two things might have occurred to the reader while doing this. First: Have we obtained too many arbitrary constant of integration? Are there really four independent such constants, A, B, C, E? Second: Might there be an easier way of finding y than the one we chose?

While pondering on these, we routinely substitute our answers back into the original differential equations just to check their correctness. (We don't need to be told to do this - we automatically do this whenever we solve any equations of any sort.)

We find (after a little algebra and calculus) that

$$\dot{x} - 3x + y = [B - A + C + (E - B)t]e^{2t}$$

and

$$\dot{y} - x - y = [E - A + C + (2C - B - E)]e^{2t}.$$

These must both be zero, which means that $E = B$ and $C = A - B$.

Thus the solutions are

$$x = (A + Bt)e^{2t}$$
$$y = (A - B + Bt)e^{2t}.$$

That was hard work!

But there is an easier way. Once we have found $x = (A + Bt)e^{2t}$, we can easily find y without any more integration merely by substitution in $\dot{x} - 3x + y = 0$, to give immediately $y = (A - B + Bt)e^{2t}$, without introducing C and E at all.

Try this pair:

$$\dot{x} - 2x + 4y = 3e^{t}$$
$$\dot{y} - x + 2y = 0$$

and this pair:

$$\ddot{x} + \dot{y} - 4y + 2y = \sin t$$
$$3\dot{x} + \ddot{y} - 6x + 4y = \cos t.$$

This latter pair is two second order equations in two variables, so you will have four independent arbitrary constants of integration. You will know whether you have the right answers by substitution back into the original equations. The solutions to the second pair are rather lengthy sums of several terms, which cannot be simplified a great deal. I mention this not because they are particularly difficult, but so that you don't waste time thinking that your answers cannot possibly be correct.