Chapter 4
Second Order Differential Equations 2
Equations of the form $ay'' + by' + cy = 0$

These are fairly straightforward. A small point to begin with: There would be no loss of generality in dividing the equation throughout by a or by b or by c, so that the coefficient of y'' or of y' or the constant is 1, thus reducing the number of parameters. The downside is that it might result in the coefficients being awkward fractions such as $\frac{4}{7}$ or something. It's a matter of choice.

If $y = e^{kx}$, then $y' = ke^{kx}$ and $y'' = k^2e^{kx}$, which immediately suggests that there is a solution of the form $y = e^{kx}$. Indeed, substitution of this into the differential equation results in $(ak^2 + bk + c)e^{kx} = 0$. That is, there is indeed a solution of the form $y = e^{kx}$, where k is a solution of the *auxiliary equation* $ak^2 + bk + c = 0$. There are two such values, (call them k_1 and k_2), so either $y = e^{k_1x}$ or $y = e^{k_2x}$ or any linear combination such as $y = Ae^{k_1x} + Be^{k_2x}$ is a solution. The last of these is *the* general solution, because there are two arbitrary constants (A and B) for a second order equation. The nature of the solution will depend (in the usual way when solving a quadratic equation) on whether $b^2 > 4ac$ or $b^2 < 4ac$ or $b^2 = 4ac$. In the case where $b^2 = 4ac$, the auxiliary equation has two equal roots, or only one distinct root, and this case will require special attention.

$b^2 > 4ac$

In this case k_1 and k_2 are real.

Examples:

i. $y'' - 5y' + 6y = 0$

The auxiliary equation is $k^2 - 5k + 6 = 0$, with solutions $k_1 = 2$, $k_2 = 3$, and the solution to the differential equation is $y = Ae^{2x} + Be^{3x}$. This may be verified by substitution in the original equation (do it!). The values of A and B are arbitrary, unless there is additional information. For example, if we are told the values of y for two values of x, A and B are then determined.

ii. $y'' + y' - 2y = 0$. The general solution is $y = Ae^{x} + Be^{-2x}$.

iii. $y'' + 7y' + 12y = 0$. The general solution is $y = Ae^{-3x} + Be^{-4x}$.

$b^2 < 4ac$

In this case k_1 and k_2 are conjugate complex numbers.

Examples

i. $y'' - 4y' + 13y = 0.$

The auxiliary equation is $k^2 - 4k + 13 = 0$, with solutions $k = 2 \pm 3i$.
The solution to the differential equation is

$$y = e^{2x}(Ae^{3ix} + Be^{-3ix}).$$

A and B are arbitrary constants − very arbitrary, in fact they may even be complex numbers. Let us write $A = A_1 + iA_2$ and $B = B_1 + iB_2$, and make use of $e^{\pm 3ix} = \cos 3x \pm i \sin 3ix$. Then the general solution is

$$y = e^{2x}\left[(A_1 + B_1)\cos 3x - (A_2 - B_2)\sin 3x + i\{(A_2 + B_2)\cos 3x + (A_1 - B_1)\sin 3x\}\right]$$

The constants A_1, A_2, B_1, B_2 are arbitrary real constants, whose values might be determined if there is any additional information. For example, in most physics problems, the variables x and y will be real, and this constraint requires that $A_2 + B_2$ and $A_1 - B_1$ are both zero. In this case the solution sought is

$$y = 2e^{2x}\left[A_1 \cos 3x - A_2 \sin 3x\right].$$

Unless there is any more information or any more constraints, the two real constants are still arbitrary, so we might as well write the solution as

$$y = e^{2x}\left[P\cos 3x + Q\sin 3x\right].$$

Now multiply and divide by $\sqrt{P^2 + Q^2}$:

$$y = e^{2x}\sqrt{P^2 + Q^2}\left[\frac{P}{\sqrt{P^2 + Q^2}}\cos 3x + \frac{Q}{\sqrt{P^2 + Q^2}}\sin 3x\right],$$

which we can now write as

$$y = Re^{2x}\sin(3x + \alpha),$$

where $R = \sqrt{P^2 + Q^2}$, $\quad \sin\alpha = \dfrac{P}{\sqrt{P^2 + Q^2}} \qquad \cos\alpha = \dfrac{Q}{\sqrt{P^2 + Q^2}}$.

The two arbitrary constants of integration are now R and α.

We see that the solution is a sinusoidal function whose amplitude increases exponentially with x.

ii. Now try: $\ddot{y} + 4\dot{y} + 13y = 0.$

Everything goes as before, except that the independent variable is t rather than x, and there is one difference in sign. The solution, predictably (assuming the y and t are real), is

$$y = Re^{-2t}\sin(3t+\alpha)$$

This is a sinusoidal wave whose amplitude rapidly decreases exponentially with time, asymptotically approaching zero. This is the equation for *damped harmonic motion*. This very important type of equation is dealt with in excruciating detail in http://orca.phys.uvic.ca/~tatum/classmechs/class11.pdf There, cases are discussed where the exponential decay time is long compared with the oscillation period (light damping), and where it is short (heavy damping).

$b^2 = 4ac$

Example: $y'' - 6y' + 9y = 0$.

The two solutions of the auxiliary equation are equal, namely $k_1 = k_2 = 3$.

A solution of the equation is $y = A_1e^{3x} + A_2e^{3x} = (A_1 + A_2)e^{3x}$. $A_1 + A_2$ can be combined into a single constant: $A_1 + A_2 = A$, so that a solution to the equation is $y = Ae^{3x}$. I say "a" solution - but it isn't the only one, because it contains only one arbitrary constant. Bearing in mind that the first and second derivatives of e^{3x} will also contain e^{3x}, we might guess that $f(x)e^{3x}$ might be a solution. Let's try something simple, such as $y = xe^{3x}$. From this, $y' = (1+3x)e^{3x}$ and $y'' = (6+9x)e^{3x}$, and Bingo! we find that $y'' - 6y' + 9y = 0$. Thus $y = xe^{3x}$ or any constant multiplier of it is indeed a solution. The most general solution is a linear combination of $y = e^{3x}$ and xe^{3x}. That is $y = Ae^{3x} + Bxe^{3x} = \underline{\underline{(A + Bx)e^{3x}}}$ is a solution, and since this includes the necessary and sufficient two arbitrary constants, this is the most general solution.

Summary
$y'' + by' + cy = 0$

$b^2 > 4c$

The general solution is $$y = Ae^{k_1x} + Be^{k_2x}$$

where k_1 and k_2 are the solutions of the auxiliary equation $k^2 + bk + c = 0$.

A and B are arbitrary - unless some further information is given.

$b^2 < 4c$

If x and y are real, the general solution is

$$y = Ae^{k_1 x}\sin(k_2 x + \alpha) \ ,$$

where the solutions to the auxiliary equation $k^2 + bk + c = 0$ are $k = k_1 \pm ik_2$, k_1 and k_2 both being real.

A and α are arbitrary - unless some further information is given.

$b^2 = 4c$

The general solution is

$$y = (A + Bx)e^{-\frac{1}{2}bx}.$$

A and B are arbitrary - unless some further information is given.