CHAPTER 8
ON THE ELECTRODYNAMICS OF MOVING BODIES

1. *Introduction*

First, I have shamelessly plagiarized the title of this chapter. I have stolen the title from that of one of the most famous physics research papers of the twentieth century – *Zur Elektrodynamik Bewegter Körper*, the paper in which Einstein described the Special Theory of Relativity in 1905. I shall be describing the motion of charged particles in electric and magnetic fields, but, unlike Einstein, I shall (unless I state otherwise – which *will* happen from time to time) be restricting the considerations of this chapter mostly to nonrelativistic speeds – that is to say speeds such that v^2/c^2 is much smaller than the level of precision one is interested in or can conveniently measure. Some relativistic aspects of electrodynamics are touched upon briefly in Chapter 15 of the Classical Mechanics notes in this series, but, apart from the fact that this chapter and Einstein's paper both deal with the motions of charged bodies in electric and magnetic fields, there will be little else in common.

Section 8.2 will deal with the motion of a charged particle in an electric field alone, and Section 8.3 will deal with the motion of a charged particle in a magnetic field alone. Section 8.4 will deal with the motion of a charged particle where both an electric and a magnetic field are present. That section may be a little more difficult than the others and may be omitted on a first reading by less experienced readers. Section 8.5 deals with the motion of a charged particle in a nonuniform magnetic field and is more difficult again.

8.2 *Charged Particle in an Electric Field*

There is really very little that can be said about a charged particle moving at nonrelativistic speeds in an electric field $\mathbf{E}$. The particle, of charge q and mass m, experiences a *force* $q\mathbf{E}$, and consequently it *accelerates* at a rate $q\mathbf{E}/m$. If it starts from rest, you can calculate how fast it is moving in time t, what distance it has travelled in time t, and how fast it is moving after it has covered a distance x, by all the usual first-year equations for uniformly accelerated motion in a straight line. If the charge is accelerated through a potential difference V, its loss of potential energy qV will equal its gain in kinetic energy $\frac{1}{2}mv^2$. Thus $v = \sqrt{2qV/m}$.

Let us calculate, using this nonrelativistic formula, the speed gained by an electron that is accelerated through 1, 10, 100, 1000, 10000, 100,000 and 1,000,000 volts, given that, for an electron, $e/m = 1.7588 \times 10^{11}$ C kg^{-1}. (The symbol for the electronic charge is usually written e. You might note here that that's a lot of coulombs per kilogram!). We'll also calculate v/c and v^2/c^2.

V volts	v m s^{-1}	v/c	v^2/c^2
1	5.931×10^5	1.978×10^{-3}	3.914×10^{-6}
10	1.876×10^6	6.256×10^{-3}	3.914×10^{-5}
100	5.931×10^6	1.978×10^{-2}	3.914×10^{-4}

1000	1.876×10^7	6.256×10^{-2}	3.914×10^{-3}
10000	5.931×10^7	1.978×10^{-1}	3.914×10^{-2}
100000	1.876×10^8	6.256×10^{-1}	3.914×10^{-1}
1000000	5.931×10^8	1.978	3.914

We can see that, even working to a modest precision of four significant figures, an electron accelerated through only a few hundred volts is reaching speeds at which v^2/c^2 is not quite negligible, and for less than a million volts, the electron is already apparently moving faster than light! Therefore for large voltages the formulas of special relativity should be used. Those who are familiar with special relativity (i.e. those who have read Chapter 15 of Classical Mechanics!), will understand that the relativistically correct relation between potential and kinetic energy is $qV = (\gamma - 1)m_0c^2$, and will be able to calculate the speeds *correctly* as in the following table. Those who are not familiar with relativity may be a bit lost here, but just take it as a warning that particles such as electrons with a very large charge-to-mass ratio rapidly reach speeds at which relativistic formulas need to be used. These figures are given here merely to give some idea of the magnitude of the potential differences that will accelerate an electron up to speeds where the relativistic formulas must be used.

V volts	v m s^{-1}	v/c	v^2/c^2
1	5.931×10^5	1.978×10^{-3}	3.914×10^{-6}
10	1.875×10^6	6.256×10^{-3}	3.914×10^{-5}
100	5.930×10^6	1.978×10^{-2}	3.912×10^{-4}
1000	1.873×10^7	6.247×10^{-2}	3.903×10^{-3}
10000	5.845×10^7	1.950×10^{-1}	3.803×10^{-2}
100000	1.644×10^8	5.482×10^{-1}	3.005×10^{-1}
1000000	2.821×10^8	0.941	0.855

If a charged particle is moving at constant speed in the x-direction, and it encounters a region in which there is an electric field in the y-direction (as in the Thomson e/m experiment, for example) it will accelerate in the y-direction while maintaining its constant speed in the x-direction. Consequently it will move in a parabolic trajectory just like a ball thrown in a uniform gravitational field, and all the familiar analysis of a parabolic trajectory will apply, except that instead of an acceleration g, the acceleration will be q/m.

8.3 *Charged Particle in a Magnetic Field*

We already know that an electric current **I** flowing in a region of space where there exists a magnetic field **B** will experience a force that is at right angles to both **I** and **B**, and the force per unit length, **F'**, is given by

$$\mathbf{F'} = \mathbf{I} \times \mathbf{B}, \qquad 8.3.1$$

and indeed we used this equation to define what we mean by **B**. Equation 8.3.1 is illustrated in figure VIII.1.

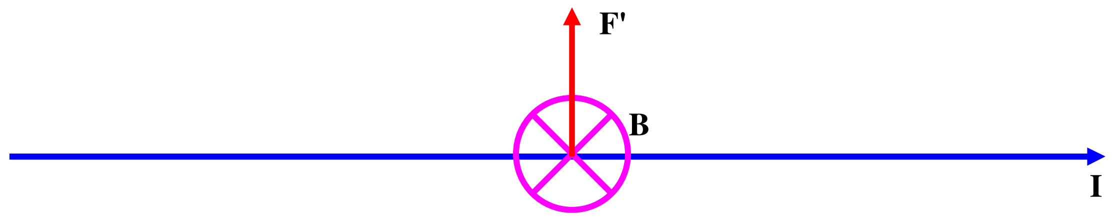


FIGURE VIII.1

The large cross in a circle is intended to indicate a magnetic field directed into the plane of the paper, and **I** and **F'** show the directions of the current and the force.

Now we might consider the current to comprise a stream of particles, *n* of them per unit length, each bearing a charge *q*, and moving with velocity **v** (speed υ). The current is then $nq\mathbf{v}$, and equation 8.3.1 then shows that the force on each particle is

$$\mathbf{F} = q\,\mathbf{v}\times\mathbf{B}\,. \qquad 8.3.2$$

This, then, is the equation that gives the force on a charged particle moving in a magnetic field, and the force is known as the *Lorentz force*.

It will be noted that there is a force on a charged particle in a magnetic field *only if the particle is moving*, and the force is at right angles to both **v** and **B.**

As to the question: "Who's to say if the particle is moving?" or "moving relative to what?" – that takes us into very deep waters indeed. For an answer, I refer you to the following paper: Einstein, A., *Zur Elektrodynamik Bewegter Körper*, Annalen der Physik **17**, 891 (1905).

Let us suppose that we have a particle, of charge *q* and mass *m*, moving with speed υ in the plane of the paper, and that there is a magnetic field **B** directed at right angles to the plane of the paper. (If you are reading this straight off the screen, then read "plane of the screen"!) The particle will experience a force of magnitude $q\upsilon B$ (because **v** and **B** are at right angles to each other), and this force is at right angles to the instantaneous velocity of the particle. Because the force is at right angles to the instantaneous velocity vector, the speed of the particle is unaffected. Its acceleration is constant in magnitude and therefore the particle moves in a *circle*, whose radius is determined by equating the force $q\upsilon B$ to the mass times the centripetal acceleration. That is $q\upsilon B = m\upsilon^2/r$, or

$$r = \frac{m\upsilon}{qB}\,. \qquad 8.3.3$$

If we are looking at the motion of some subatomic particle in a magnetic field, and we have reason to believe that the charge is equal to the electronic charge (or perhaps some small multiple of it), we see that the radius of the circular path tells us the *momentum* of the particle; that is, the product of its mass and speed. Equation 8.3.3 is quite valid for relativistic speeds, except that the mass that appears in the equation is then the relativistic mass, not the rest mass, so that the radius is a slightly more complicated function of speed and rest mass.

If **v** and **B** are not perpendicular to each other, we may resolve **v** into a component υ_1 perpendicular to **B** and a component υ_2 parallel to **B**. The particle will then move in a *helical* path, the radius of the helix being $m\upsilon_2/(qB)$, and the centre of the circle moving at speed υ_2 in the direction of **B**.

The angular speed ω of the particle in its circular path is $\omega = \upsilon / r$, which, in concert with equation 8.3.3, gives

$$\omega = \frac{qB}{m}. \qquad 8.3.4$$

This is called the *cyclotron angular speed* or the *cyclotron angular frequency.* You should verify that its dimensions are T^{-1}.

A *magnetron* is an evacuated cylindrical glass tube with two electrodes inside. One, the negative electrode (cathode) is a wire along the axis of the cylinder. This is surrounded by a hollow cylindrical anode of radius a. A uniform magnetic field is directed parallel to the axis of the cylinder. The cathode is heated (and emits electrons, of charge e and mass m) and a potential difference V is established across the electrodes. The electrons consequently reach a speed given by

$$eV = \tfrac{1}{2}m\upsilon^2. \qquad 8.3.5$$

Because of the magnetic field, they move in arcs of circles. As the magnetic field is increased, the radius of the circles become smaller, and, when the diameter of the circle is equal to the radius a of the anode, no electrons can reach the anode, and the current through the magnetron suddenly drops. This happens when

$$\tfrac{1}{2}a = \frac{m\upsilon}{eB}. \qquad 8.3.6$$

Elimination of υ from equations 8.3.5 and 8.3.6 shows that the current drops to zero when

$$B = \sqrt{\frac{8mV}{ea^2}}. \qquad 8.3.7$$

Those who are skilled in special relativity should try and do this with the relativistic formulas. In equation 8.3.5 the right hand side will have to be $(\gamma - 1)m_0c^2$, and in equation 8.3.6 m will have to be replaced with γm_0. I make the result

$$B = \frac{2\sqrt{2m_0c^2eV + e^2V^2}}{eac}. \qquad 8.3.8$$

For small potential differences, eV is very much less than m_0c^2, and equation 8.3.8 reduces to equation 8.3.5.

8.4 *Charged Particle in an Electric and a Magnetic Field*

The force on a charged particle in an electric and a magnetic field is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \qquad 8.4.1$$

As an example, let us investigate the motion of a charged particle in uniform electric and magnetic fields that are at right angles to each other. Specifically, let us choose axes so that the magnetic field **B** is directed along the positive z-axis and the electric field is directed along the positive y-axis. (Draw this on a large diagram!) Try and imagine what the motion would be like. Suppose, for example, the motion is all in the yz-plane. Perhaps the particle will move round and round in a circle around an axis parallel to the magnetic field, but the centre of this circle will accelerate in the direction of the electric field. Well, you are right in that the particle does move in a circle around an axis parallel to **B**, and also that the centre of the circle does indeed move. But the rest of it isn't quite right. Before embarking on a mathematical analysis, see if you can imagine the motion a bit more accurately.

We'll suppose that at some instant the x, y and z components of the velocity of the particle are u , v and w. We'll suppose that these velocity components are all nonrelativistic, which means that m is constant and not a function of the speed. The three components of the equation of motion (equation 8.4.1) are then

$$m\dot{u} = qBv\,, \qquad 8.4.2$$

$$m\dot{v} = -qBu + qE \qquad 8.4.3$$

and $$m\dot{w} = 0. \qquad 8.4.4$$

For short, I shall write $q\,B/m = \omega$ (the cyclotron angular speed) and, noting that the dimensions of E/B are the dimensions of speed (verify this!), I shall write $E/B = V_\mathrm{D}$, where the significance of the subscript D will become apparent in due course. The equations of motion then become

$$\ddot{x} = \dot{u} = \omega v, \qquad 8.4.5$$

$$\ddot{y} = \dot{v} = -\omega(u - V_\mathrm{D}) \qquad 8.4.6$$

and $$\ddot{z} = \dot{w} = 0. \qquad 8.4.7$$

To find the general solutions to these, we can, for example, let $X = u - V_\mathrm{D}$. Then equations 8.4.5 and 8.4.6 become $\dot{X} = \omega v$ and $\dot{v} = -\omega X$. From these, we obtain $\ddot{X} = -\omega^2 X$. The general solution of this is $X = A\sin(\omega t + \alpha)$, and so $u = A\sin(\omega t + \alpha) + V_{\rm D}$. By integration and differentiation with respect to time we can find x and $\ddot{x}$ respectively. Thus we obtain:

$$x = -\frac{A}{\omega}\cos(\omega t + \alpha) + V_{\rm D}t + D, \qquad 8.4.8$$

$$u = \dot{x} = A\sin(\omega t+\alpha) + V_{\rm D} \qquad 8.4.9$$

and

$$\ddot{x} = A\omega\cos(\omega t+\alpha). \qquad 8.4.10$$

Similarly we can solve for y and z as follows:

$$y = \frac{A}{\omega}\sin(\omega t + \alpha) + F, \qquad 8.4.11$$

$$\upsilon = \dot{y} = A\cos(\omega t+\alpha), \qquad 8.4.12$$

$$\ddot{y} = -A\omega\sin(\omega t+\alpha), \qquad 8.4.13$$

$$z = w_0 t + z_0, \qquad 8.4.14$$

$$w = \dot{z} = w_0 \qquad 8.4.15$$

and

$$\ddot{z} = 0. \qquad 8.4.16$$

There are six arbitrary constants of integration, namely A, D, F, α, z_0 and w_0, whose values depend on the initial conditions (position and velocity at $t = 0$). Of these, z_0 and w_0 are just the initial values of z and w. Let us suppose that these are both zero and that all the motion takes place in the xy-plane.

In these equations A and α always occur in the combinations $A \sin \alpha$ and $A \cos \alpha$, and therefore for convenience I am going to let $A \sin \alpha = S$ and $A \cos \alpha = C$, and I am going to re-write equations 8.4.8, 8.4.9, 8.4.11 and 8.4.12 as

$$x = -\frac{1}{\omega}(C\cos\omega t - S\sin\omega t) + V_{\rm D}t + D, \qquad 8.4.17$$

$$u = C\sin\omega t + S\cos\omega t + V_{\rm D}, \qquad 8.4.18$$

$$y = \frac{1}{\omega}(C\sin\omega t + S\cos\omega t) + F, \qquad 8.4.19$$

and

$$\upsilon = C\cos\omega t - S\sin\omega t. \qquad 8.4.20$$

Let is suppose that the initial conditions are: at $t = 0$, $x = y = u = \upsilon = 0$. That is, the particle starts from rest at the origin. If the put these initial conditions in equations 8.4.17-20, we find that $C = 0$, $S = -V_D$, $D = 0$ and $F = V_D/\omega$. Equations 8.4.17 and 8.4.19, which give the equation to the path described by the particle, become

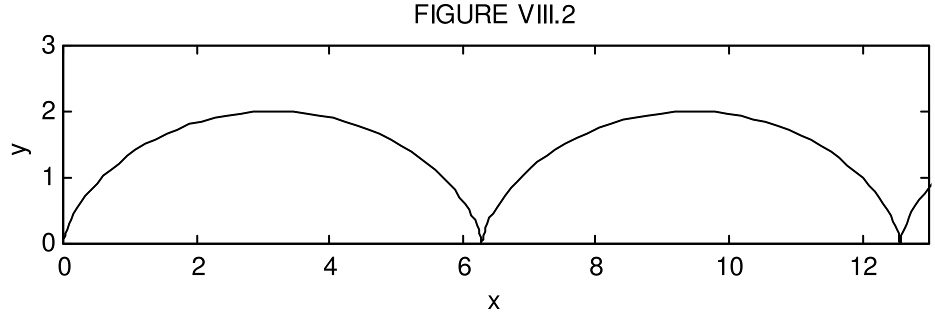
$$x = -\frac{V_D}{\omega}\sin\omega t + V_D t \qquad 8.4.21$$

and

$$y = \frac{V_D}{\omega}(1 - \cos\omega t). \qquad 8.4.22$$

It is worth reminding ourselves here that the cyclotron angular speed is $\omega = qB/m$ and that $V_D = E/B$, and therefore $\frac{V_D}{\omega} = \frac{mE}{qB^2}$. These equations are the parametric equations of a *cycloid*. (For more on the cycloid, see Chapter 19 of the Classical Mechanics notes in this series.) The motion is a circular motion in which the centre of the circle *drifts* (hence the subscript D) in the x-direction at speed V_D. The path is shown in figure VIII.2, drawn for distances in units of $\frac{V_D}{\omega} = \frac{mE}{qB^2}$.

FIGURE VIII.2



I leave it to the reader to try different initial conditions, such as one of u or υ *not* initially zero. You can try with u_0 or υ_0 equal to some multiple of fraction of V_D, and you can make the u_0 or υ_0 positive or negative. Calculate the values of the constants D, F, C and S and draw the resulting path. You will always get some sort of cycloid. It may not be a simple cycloid as in our example, but it might be an *expanded cycloid* (i.e. small loops instead of cusps) or a *contracted cycloid*, which has neither loops nor cusps, but looks more or less sinusoidal. I'll try just one. I'll let $u_0 = 0$ and $\upsilon_0 = +V_D$. If I do that, I get

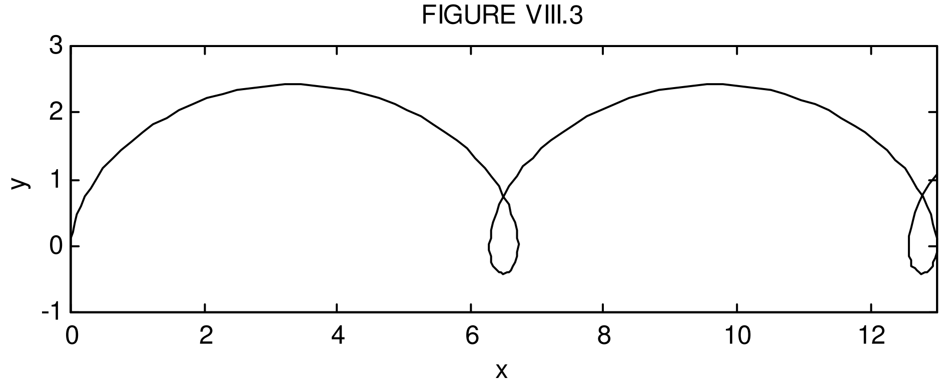
$$x = \frac{V_D}{\omega}(1 - \cos\omega t - \sin\omega t) + V_D t \qquad 8.4.23$$

and

$$y = \frac{V_D}{\omega}(1 - \cos\omega t + \sin\omega t). \qquad 8.4.24$$

This looks like this:

FIGURE VIII.3



8.5 *Motion in a Nonuniform Magnetic Field*

I give this as a rather more difficult example, not suitable for beginners, just to illustrate how one might calculate the motion of a charged particle in a magnetic field that is not uniform. I am going to suppose that we have an electric current I flowing (in a wire) in the positive z-direction up the z-axis. An electron of mass m and charge of magnitude e (i.e., its charge is $-e$) is wandering around in the vicinity of the current. The current produces a magnetic field, and consequently the electron, when it moves, experiences a Lorenz force. In the following table I write, in cylindrical coordinates, the components of the magnetic field produced by the current, the components of the Lorentz force on the electron, and the expressions in cylindrical coordinates for acceleration component. Some facility in classical mechanics will be needed to follow this.

	Field	Force	Acceleration
ρ	$B_\rho = 0$	$e\dot{z}B_\phi$	$\ddot{\rho} - \rho\dot{\phi}^2$
ϕ	$B_\phi = \dfrac{\mu_0 I}{2\pi\rho}$	0	$\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}$
z	$B_z = 0$	$-e\dot{\rho}B_\phi$	$\ddot{z}$

From this table we can write down the *equations of motion*, as follows, in which S_C is short for $\dfrac{\mu_0 eI}{2\pi m}$. This quantity has the dimensions of *speed* (verify!) and I am going to call it the *characteristic speed.* It has the numerical value $3.5176 \times 10^4\ I\ \mathrm{m\ s^{-1}}$, where I is in A. The equations of motion, then, are

Radial: $$\rho(\ddot{\rho} - \rho\dot{\phi}^2) = S_\mathrm{C}\dot{z} \qquad 8.5.1$$

Transverse (Azimuthal): $$\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi} = 0 \qquad 8.5.2$$

Longitudinal: $$\rho\ddot{z} = -S_\mathrm{C}\dot{\rho}. \qquad 8.5.3$$

It will be convenient to define dimensionless velocity components:

$$u = \dot{\rho}/S_\mathrm{C}, \quad \upsilon = \rho\dot{\phi}/S_\mathrm{C}, \quad w = \dot{z}/S_\mathrm{C}. \qquad 8.5.4\mathrm{a,b,c}$$

Suppose that initially, at time $t = 0$, their values are u_0, υ_0 and w_0, and also that the initial distance of the particle from the current is ρ_0. Further, introduce the dimensionless distance

$$x = \rho/\rho_0, \qquad 8.5.5$$

so that the initial value of x is 1. The initial values of ϕ and z may be taken to be zero by suitable choice of axes.

Integration of equations 8.5.2 and 3, with these initial conditions, yields

$$\dot{z} = S_C\left(w_0 - \ln(\rho/\rho_0)\right) \qquad 8.5.6$$

and

$$\rho^2\dot{\phi} = \rho_0 \upsilon_0 S_C\,; \qquad 8.5.7$$

or, in terms of the dimensionless variables,

$$w = w_0 - \ln x \qquad 8.5.8$$

and

$$\upsilon = \upsilon_0 / x. \qquad 8.5.9$$

We may write $\dot{\rho}\dfrac{d\dot{\rho}}{d\rho}$ for $\ddot{\rho}$ in equation 8.5.1, and substitution for $\dot{z}$ and $\dot{\phi}$ from equations 8.5.6 and 8.5.7 yields

$$u^2 = u_0^2 + \upsilon_0^2(1 - 1/x^2) + 2w_0 \ln x - (\ln x)^2. \qquad 8.5.10$$

Equations 8.5.8,9 and 10 give the velocity components of the electron as a function of its distance from the wire.

Equation 8.5.2 expresses the fact that there is no transverse (azimuthal) force. Its time integral (equation 8.5.7) expresses the consequence that the z-component of its angular momentum is conserved. Further, from equations 8.5.8,9 and 10, we find that

$$u^2 + \upsilon^2 + w^2 = u_0^2 + \upsilon_0^2 + w_0^2 = s^2, \text{ say,} \qquad 8.5.11$$

so that the speed of the electron is constant. This is as expected, since the force on the electron is always perpendicular to its velocity; the point of applicat6ion of the force does not move in the direction of the force, which therefore does no work, so that kinetic energy, and hence speed, is conserved.

The distance of the electron from the wire is bounded below and above. The lower and upper bounds, x_1 and x_2 are found from equation 8.5.10 by putting $u = 0$ and solving for x. Examples of these bounds are shown in the Table VIII.I for a variety of initial conditions.

TABLE VIII.1

BOUNDS OF THE MOTION

$\lvert u_0 \rvert$	$\lvert v_0 \rvert$	$\lvert w_0 \rvert$	x_1	x_2
0	0	–2	0.018	1.000
0	0	–1	0.135	1.000
0	0	0	1.000	1.000
0	0	1	1.000	7.389
0	0	2	1.000	54.598
0	1	–2	0.599	1.000
0	1	–1	1.000	1.000
0	1	0	1.000	2.501
0	1	1	1.000	11.149
0	1	2	1.000	69.132
0	2	–2	1.000	1.845
0	2	–1	1.000	3.137
0	2	0	1.000	7.249
0	2	1	1.000	25.398
0	2	2	1.000	125.009
1	0	–2	0.014	1.266
1	0	–1	0.089	1.513
1	0	0	0.368	2.718
1	0	1	0.661	11.181
1	0	2	0.790	69.135
1	1	–2	0.476	1.412
1	1	–1	0.602	1.919
1	1	0	0.726	4.024
1	1	1	0.809	15.345
1	1	2	0.857	85.581
1	2	–2	0.840	2.420
1	2	–1	0.873	4.052
1	2	0	0.896	9.259
1	2	1	0.912	31.458
1	2	2	0.925	148.409
2	0	–2	0.008	2.290
2	0	–1	0.039	3.442
2	0	0	0.135	7.389
2	0	1	0.291	25.433
2	0	2	0.437	125.014
2	1	–2	0.352	2.654
2	1	–1	0.409	4.212
2	1	0	0.474	9.332
2	1	1	0.542	31.478

$\lvert u_0 \rvert$	$\lvert \upsilon_0 \rvert$	$\lvert w_0 \rvert$	x_1	x_2
2	1	2	0.605	148.412
2	2	−2	0.647	4.183
2	2	−1	0.681	7.297
2	2	0	0.712	16.877
2	2	1	0.740	54.486
2	2	2	0.764	236.061

In analysing the motion in more detail, we can start with some particular initial conditions. One easy case is if $u_0 = \upsilon_0 = w_0 = 0$ – i.e. the electron starts at rest. In that case there will be no forces on it, and it remains at rest for all time. A less trivial initial condition is for $\upsilon_0 = 0$, but the other components not zero. In that case, equation 8.5.7 shows that ϕ is constant for all time. What this means is that the motion all takes place in a plane ϕ = constant, and there is no motion "around" the wire. This is just to be expected, because the ρ-component of the velocity gives rise to a z-component of the Lorenz force, and the z-component of the velocity gives rise to a Lorentz force towards the wire, and there is no component of force "around" (increasing ϕ) the wire. The electron, then, is going to move in the plane ϕ = constant at a constant speed $S = sS_C$, where $s = \sqrt{u_0^2 + w_0^2}$. (Recall that u and w are dimensionless quantities, being the velocity components *in units of the characteristic speed* S_C.) I am going to coin the words *perineme* and *aponeme* to describe the least and greatest distances of the electrons from the wire – i.e. the bounds of the motion. These bounds can be found by setting $u = 0$ and $\upsilon_0 = 0$ in equation 8.5.10 (where we recall that $x = \rho/\rho_0$ - i.e. the ratio of the radial distance of the electron at some time to its initial radial distance). We obtain

$$\rho = \rho_0 e^{w_0 \pm s} \qquad 8.5.12$$

for the aponeme (upper sign) and perineme (lower sign) distances. From equation 8.5.8 we can deduce that the electron is moving at right angles to the wire (i.e. $w = 0$) when it is at a distance

$$\rho = \rho_0 e^{w_0}. \qquad 8.5.13$$

The form of the trajectory with $\upsilon_0 = 0$ is found by integrating equations 8.5.8 and 8.5.10. It is convenient to start the integration at perineme so that $u_0 = 0$ and $s = w_0$, and the initial value of x $(= \rho/\rho_0)$ is 1. For any other initial conditions, the perineme values of x and ρ can be found from equations 8.5.10 and 8.5.12 respectively. Equations 8.5.10 and 8.5.8 may them be written

$$t = \frac{\rho_0}{S_C}\int_1^x \frac{dx}{[2s\ln x - (\ln x)^2]^{1/2}} \qquad 8.5.14$$

and $$z = St - \rho_0 \int_1^x \frac{\ln x\, dx}{[2s\ln x - (\ln x)^2]^{1/2}}\,. \qquad 8.5.15$$

There are singularities in the integrands at $x = 1$ and $\ln x = 2s$, and, in order to circumvent this difficulty it is convenient to introduce a variable θ defined by

$$\ln x = s(1 - \sin\theta). \qquad 8.5.16$$

Equations 8.5.14 and 15 then become

$$t = \frac{\rho_0 e^s}{S_C}\int_{\pi/2}^{\theta} e^{-s\sin\theta} d\theta \qquad 8.5.17$$

and $$z = \rho_0 s e^s \int_{\pi/2}^{\theta} \sin\theta\, e^{-s\sin\theta} d\theta. \qquad 8.5.18$$

Examples of these trajectories are shown in figure VIII.4, though I'm afraid you will have to turn your monitor on its side to view it properly. They are drawn for s = 0.25, 0.50, 1.00 and 2.00, where s is the ratio of the constant electron speed to the characteristic speed S_C. The wire is supposed to be situated along the z-axis ($\rho = 0$) with the current flowing in the direction of positive z. The electron drifts in the opposite direction to the current. (A positively charged particle would drift in the same direction as the current.) Distances in the figure are expressed in terms of the perineme distance ρ_0. The shape of the path depends only on s (and not on ρ). For no speed does the path have a cusp. The radius of curvature R at any point is given by $R = \rho/s$.

Minima of ρ occur at $\rho = \rho_0$ and $\theta = (4n + 1)\pi/2$, where n is an integer;

Maxima of z occur at $\rho = \rho_0 e^s$ and $\theta = (4n + 2)\pi/2$;

Maxima of ρ occur at $\rho = \rho_0 e^{2s}$ and $\theta = (4n + 3)\pi/2$;

Minima of z occur at $\rho = \rho_0 e^s$ and $\theta = (4n + 4)\pi/2$.

The distance between successive loops and the period of each loop vary rapidly with electron speed, as is illustrated in Table VIII.2. In this table, s is the electron speed in units of the characteristic speed S_C, A_1 is the ratio of aponeme to perineme distance, A_2 is the ratio of interloop distance to perineme distance, A_3 is the ratio of period per loop to ρ_0/S_C, and A_4 is the drift speed in units of the characteristic speed S_C .

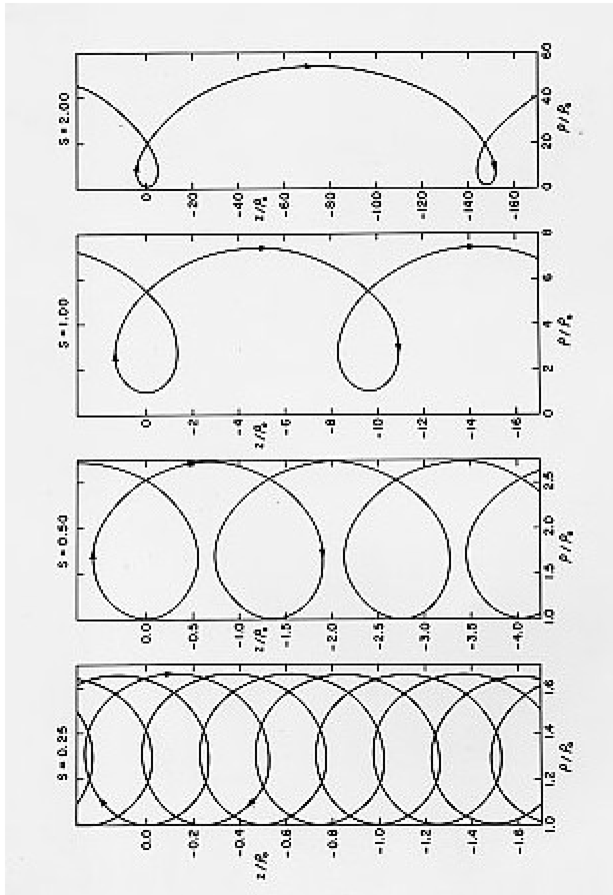


FIGURE VIII.4

For example, for a current of 1 A, the characteristic speed is 3.5176×10^4 m s^{-1}. If an electron is accelerated through 8.7940 V, it will gain a speed of 1.7588 m s^{-1}, which is 50 times the characteristic speed. If the electron starts off at this speed moving in the same direction as the current and 1 Å (10^{-10} m) from it, it will reach a maximum distance of 8.72×10^{10} megaparsecs (1 Mpc = 3.09×10^{22} m) from it, provided the Universe is euclidean. The distance between the loops will be 1.53×10^{12} Mpc, and the period will be 8.60×10^{20} years, after which the electron will have covered, at constant speed, a total distance of 1.55×10^{12} Mpc. The drift speed will be 1.741×10^6 m s^{-1}.

TABLE VIII.2

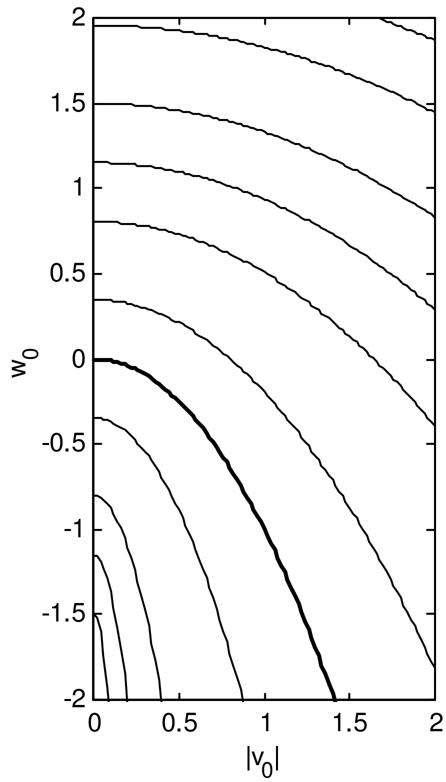
s	A_1	A_2	A_3	A_4
0.1	1.22	3.47×10^{-2}	6.96	4.99×10^{-3}
0.2	1.49	1.54×10^{-1}	7.75	1.99×10^{-2}
0.5	2.72	1.34	11.0	0.122
1.0	7.39	9.65	21.6	0.447
2.0	54.6	1.48×10^{2}	1.06×10^{2}	1.40
5.0	2.20×10^{4}	1.13×10^{5}	2.54×10^{4}	4.49
10.0	4.85×10^{8}	3.70×10^{9}	3.90×10^{8}	9.49
20.0	2.35×10^{17}	2.59×10^{18}	1.33×10^{17}	19.5
50.0	2.69×10^{43}	4.73×10^{44}	9.55×10^{42}	49.5

Let us now turn to consideration of cases where $\upsilon_0 \neq 0$, so that the motion of the electron is not restricted to a plane. At first glance is might be thought that since an azimuthal velocity component gives rise to no additional Lorenz force on the electron, the motion will hardly be affected by a nonzero υ_0, other than perhaps by a revolution around the wire. In particular, for given initial velocity components u_0 and w_0, the perineme and aponeme distances x_1 and x_2 might seem to be independent of υ_0 . Reference to Table VIII.1, however, shows that this is by no means so. The reason is that as the electron moves closer to or further from the wire, the changes in υ made necessary by conservation of the z-component of the angular momentum are compensated for by corresponding changes in u and w made necessary by conservation of kinetic energy.

Since the motion is bounded above and below, there will always be some time when $\dot{\rho} = 0$. There is no loss of generality if we shift the time origin so as to choose $\dot{\rho} = 0$ when $t = 0$ and $x = 1$. From this point, therefore, we shall consider only those trajectories for which $u_0 = 0$. In other words we shall follow the motion from a time $t = 0$ when the electron is at an apsis ($\dot{\rho} = 0$). [The plural of *apsis* is *apsides*. The word *apse* (plural *apses*) is often used in this connection, but it seems useful to maintain a distinction between the architectural term *apse* and the mathematical term *apsis.*] Whether this apsis is perineme (so that $\dot{\rho} = \rho_1$, $\upsilon_0 = \upsilon_1$, $w_0 = w_1$) or aponeme (so that $\dot{\rho} = \rho_2$, $\upsilon_0 = \upsilon_2$, $w_0 = w_2$) depends on the subsequent motion.

The electron starts, then, at a distance from the wire defined by $x = 1$. It is of interest to find the value of x at the next apsis, in terms of the initial velocity components υ_0 and w_0. This is found from equation 8.5.10 with u = 0 and u_0 = 0. The results are shown in figure VIII.4. This figure shows loci of constant next apsis distance, for values of x (going from bottom left to top right of the figure) of 0.05, 0.10, 0.20, 0.50, 1, 2, 5, 10, 20, 50, 100. The heavy curve is for x = 1. It will immediately be seen that, if $w_0 > -\upsilon_0^2$, (above the heavy curve) the value of x at the second apsis is greater than 1. (Recall that υ and w are dimensionless ratios, so there is no problem of dimensional imbalance in the inequality.) The electron was therefore initially at perineme and subsequently moves away from the wire. If on the other hand $w_0 < -\upsilon_0^2$, (below the heavy curve) the value of x at the second apsis is less than 1. The electron was therefore initially at aponeme and subsequently moves closer to the wire.

FIGURE VIII. 5



The case where $w_0 = -\upsilon_0^2$ is of special interest, for then perineme and aponeme distances are equal and indeed the electron stays at a constant distance from the wire at all times. It moves in a helical trajectory drifting in the opposite direction to the direction of the conventional current I. (A positively charged particle would drift in the same direction as I.) The pitch angle α of the helix (i.e. the angle between the instantaneous velocity and a plane normal to the wire) is given by

$$\tan\alpha = -w/\upsilon, \qquad 8.5.19$$

where w and υ are constrained by the equations

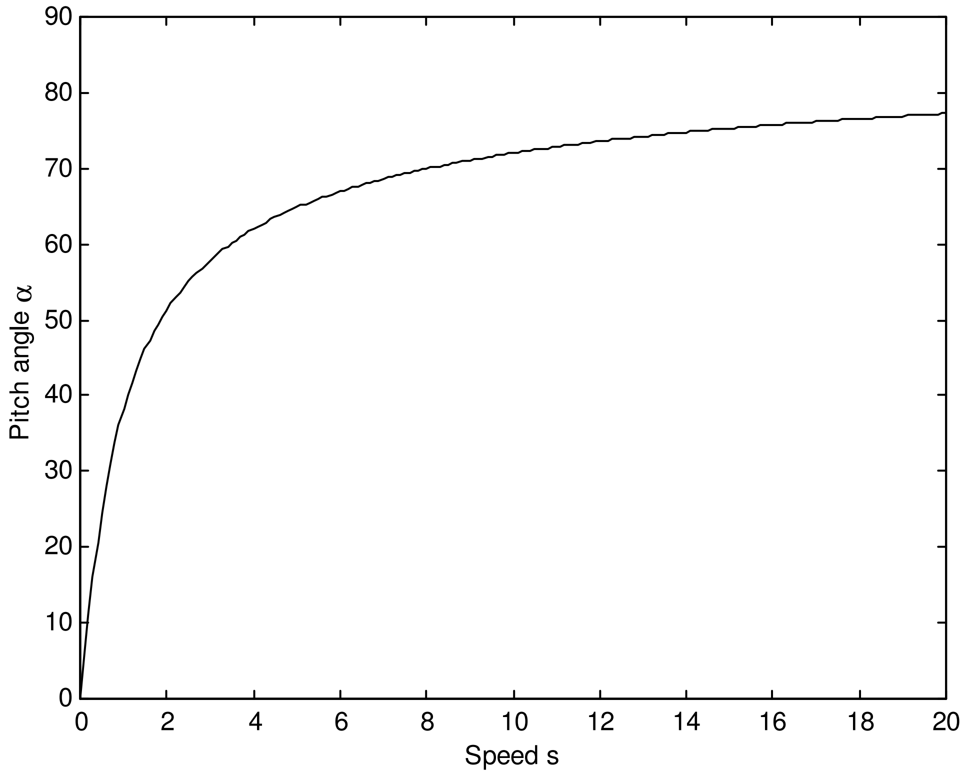
$$w_0 = -\upsilon_0^2 \qquad 8.5.20$$

and

$$\upsilon^2 + w^2 = s^2. \qquad 8.5.21$$

This implies that the pitch angle is determined solely by s, the ratio of the speed S of the electron to the characteristic speed S_C. On other words, the pitch angle is determined by the ratio of the electron speed S to the current I. The variation of pitch angle α with speed s is shown in figure VIII.6. This relation is entirely independent of the radius of the helix.

FIGURE VIII 6



If $w_0 \neq -\upsilon_0^2$, the electron no longer moves in a simple helix, and the motion must be calculated numerically for each case. It is convenient to start the calculation at perineme with initial conditions $u_0 = 0,\ w_0 > -\upsilon_0^2,\ x_0 = 1$. For other initial conditions, the perineme (and aponeme) values of u, υ, w and ρ can easily be found from equations 8.5.10 (with $u_0 = 0$), 8.5.8 and 8.5.9. Starting, then, from perineme, integrations of these equations take the respective forms

$$t = \frac{\rho_0}{S_C}\int_1^x [\upsilon_0^2(1 - 1/x^2) + 2w_0 \ln x - (\ln x)^2]^{-1/2} dx, \qquad 8.5.22$$

$$z \;=\; w_0 S_C t - \rho_0 \int_1^x [\upsilon_0^2 (1 - 1/x^2) \;+\; 2w_0 \ln x \;-\; (\ln x)^2]^{-1/2} \ln x \; dx \qquad 8.5.23$$

and

$$\phi \;=\; \upsilon_0 \int_1^x [\upsilon_0^2 (1 - 1/x^2) \;+\; 2w_0 \ln x \;-\; (\ln x)^2]^{-1/2} x^{-2} \, dx. \qquad 8.5.24$$

The integration of these equations is not quite trivial and is discussed in the Appendix (Section 8A).

In general the motion of the electron can be described qualitatively roughly as follows. The motion is bounded between two cylinders of radii equal to the perineme and aponeme distances, and the speed is constant. The electron moves around the wire in either a clockwise or a counterclockwise direction, but, once started, the sense of this motion does not change. The angular speed around the wire is greatest at perineme and least at aponeme, being inversely proportional to the square of the distance from the wire. Superimposed on the motion around the wire is a general drift in the opposite direction to that of the conventional current. However, for a brief moment near perineme the electron is temporarily moving in the same direction as the current.

An example of the motion is given in figures VIII.7 and 8 for initial velocity components $u_0 = 0$, $\upsilon_0 = w_0 = 1$. The aponeme distance is 11.15 times the perineme distance. The time interval between two perineme passages is 26.47 ρ_0/S_C . The time interval for a complete revolution around the wire ($\phi = 360^\circ$) is 68.05 ρ_0/S_C . In figure VIII.8, the conventional electric current is supposed to be flowing into the plane of the "paper" (computer screen), away from the reader. The portions of the electron trajectory where the electron is moving towards from the reader are drawn as a continuous line, and the brief portions near perineme where the electron is moving away from the reader are indicated by a dotted line. Time marks on the figure are at intervals of $5\rho_0/S_C$.

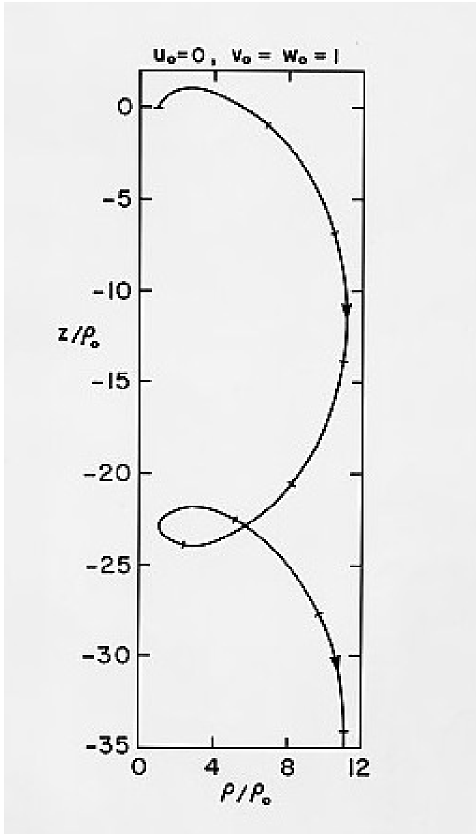


FIGURE VIII.7

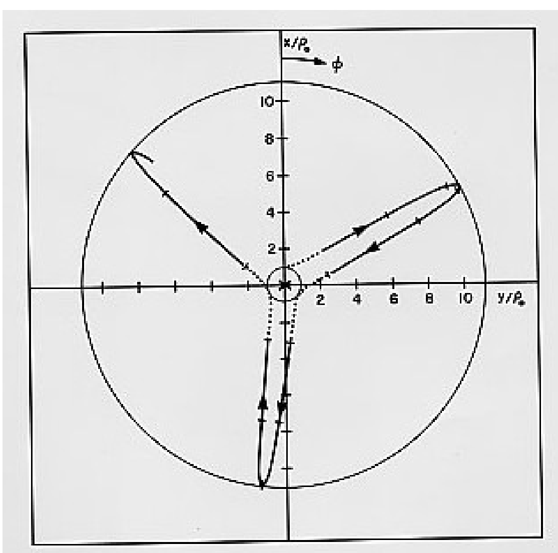


FIGURE VIII.8

8A *Appendix. Integration of the Equations*

Numerical integration of equations 8.5.22-24 is straightforward (by Simpson's rule, for example) except near perineme (x = 1) and aponeme ($x = x_2$), where the integrands become infinite. Near perineme, however, we can substitute $x = 1 + \xi$ and near aponeme we can substitute $x = x_2(1 - \xi)$, and we can expand the integrands as power series in ξ and integrate term by term. I gather here the following results for the intervals x = 1 to $x = 1 + \varepsilon$ and $x = x_2 - \varepsilon$ to $x = x_2$, where ε must be chosen to be sufficiently small that ε^4 is smaller than the precision required.

$$I_1 = \int_1^{1+\varepsilon} [\upsilon_0^2(1 - 1/x^2) + 2w_0 \ln x - (\ln x)^2]^{-1/2} dx = M(1 + \tfrac{1}{3}A_1\varepsilon + \tfrac{1}{5}B_1\varepsilon^2 + \tfrac{1}{7}C_1\varepsilon^3 + \ldots) \qquad \text{8A.1}$$

$$I_2 = \int_1^{1+\varepsilon} [\upsilon_0^2(1 - 1/x^2) + 2w_0 \ln x - (\ln x)^2]^{-1/2} \ln x dx = M(\tfrac{1}{3}\varepsilon + \tfrac{1}{5}D_1\varepsilon^2 + \tfrac{1}{7}E_1\varepsilon^3 + \ldots) \qquad \text{8A.2}$$

$$I_3 = \int_1^{1+\varepsilon} [\upsilon_0^2(1 - 1/x^2) + 2w_0 \ln x - (\ln x)^2]^{-1/2} x^{-2} dx = M(1 + \tfrac{1}{3}F_1\varepsilon + \tfrac{1}{5}G_1\varepsilon^2 + \tfrac{1}{7}H_1\varepsilon^3 + \ldots) \qquad \text{8A.3}$$

$$I_4 = \int_{x_2-\varepsilon}^{x_2} [\upsilon_0^2(1 - 1/x^2) + 2w_0 \ln x - (\ln x)^2]^{-1/2} dx = N[1 + \tfrac{1}{3}A_2\varepsilon/x_2 + \tfrac{1}{5}B_2(\varepsilon/x_2)^2 + \tfrac{1}{7}C_2(\varepsilon/x_2)^3 + \ldots) \qquad \text{8A.4}$$

$$I_5 = \int_{x_2-\varepsilon}^{x_2} [\upsilon_0^2(1 - 1/x^2) + 2w_0 \ln x - (\ln x)^2]^{-1/2} \ln x dx$$

$$= I_4 \ln x_2 - N[\tfrac{1}{3}\varepsilon/x_2 + \tfrac{1}{5}D_2(\varepsilon/x_2)^2 + \tfrac{1}{7}E_2(\varepsilon/x_2)^3 + \ldots] \qquad \text{8A.5}$$

$$I_6 = \int_{x_2-\varepsilon}^{x_2} [\upsilon_0^2(1 - 1/x^2) + 2w_0 \ln x - (\ln x)^2]^{-1/2} x^{-2} dx$$

$$= N[1 + \tfrac{1}{3}F_2\varepsilon/x_2 + \tfrac{1}{5}G_2(\varepsilon/x_2)^2 + \tfrac{1}{7}H_2(\varepsilon/x_2)^3 + \ldots]/x_2^2 \qquad \text{8A.6}$$

The constants are defined as follows.

$$M = \left(\frac{2\varepsilon}{\upsilon_0^2 + w_0}\right)^{1/2} \qquad \text{8A.7}$$

$$N = \left(\frac{2\varepsilon x_2}{\ln x_2 - (\upsilon_0/x_2)^2 - w_0}\right)^{1/2} \qquad \text{8A.8}$$

$$a_1 = -\frac{3\upsilon_0^2 + w_0 + 1}{2(\upsilon_0^2 + w_0)} \qquad \text{8A.9}$$

$$b_1 = \frac{4\upsilon_0^2 + \tfrac{2}{3}w_0 + 1}{2(\upsilon_0^2 + w_0)} \qquad \text{8A.10}$$

$$c_1 = -\frac{5\upsilon_0^2 + \frac{1}{2}w_0 + \frac{11}{12}}{2(\upsilon_0^2 + w_0)} \qquad \text{8A.11}$$

$$a_2 = \frac{3(\upsilon_0/x_2)^2 + w_0 - \ln x_2 + 1}{2\left((\upsilon_0/x_2)^2 + w_0 - \ln x_2\right)} \qquad \text{8A.12}$$

$$b_2 = \frac{4(\upsilon_0/x_2)^2 + \frac{2}{3}w_0 - \ln x_2 + 1}{2\left((\upsilon_0/x_2)^2 + w_0 - \ln x_2\right)} \qquad \text{8A.13}$$

$$c_2 = \frac{5(\upsilon_0/x_2)^2 + \frac{1}{2}w_0 - \frac{1}{2}\ln x_2 + \frac{11}{12}}{2\left((\upsilon_0/x_2)^2 + w_0 - \ln x_2\right)} \qquad \text{8A.14}$$

$$A_n = -\tfrac{1}{2}a_n \qquad \text{8A.15}$$

$$B_n = -\tfrac{1}{2}b_n + \tfrac{3}{8}a_n^2 \qquad \text{8A.16}$$

$$C_n = -\tfrac{1}{2}c_n + \tfrac{3}{4}a_n b_n - \tfrac{5}{16}a_n^3 \qquad \text{8A.17}$$

$$D_n = A_n + \tfrac{1}{2}(-1)^n \qquad \text{8A.18}$$

$$E_n = B_n + \tfrac{1}{2}(-1)^n A_n + \tfrac{1}{3} \qquad \text{8A.19}$$

$$F_n = A_n + 2(-1)^n \qquad \text{8A.20}$$

$$G_n = B_n + 2(-1)^n A_n + 3 \qquad \text{8A.21}$$

$$H_n = C_n + 2(-1)^n B_n + 3A_n + 4(-1)^n \qquad \text{8A.22}$$

$n = 1, 2$