CHAPTER 2
ELECTROSTATIC POTENTIAL

2.1 *Introduction*

Imagine that some region of space, such as the room you are sitting in, is permeated by an electric field. (Perhaps there are all sorts of electrically charged bodies outside the room.) If you place a small positive test charge somewhere in the room, it will experience a force $\mathbf{F} = Q\mathbf{E}$. If you try to move the charge from point A to point B against the direction of the electric field, you will have to do work. If work is required to move a positive charge from point A to point B, there is said to be an electrical *potential difference* between A and B, with point A being at the lower potential. If one joule of work is required to move one coulomb of charge from A to B, the potential difference between A and B is one *volt* (V).

The dimensions of potential difference are $\mathrm{ML^2T^{-2}Q^{-1}}$.

All we have done so far is to define the potential *difference* between *two points*. We cannot define "the" potential at *a point* unless we arbitrarily assign some reference point as having a defined potential. It is not always necessary to do this, since we are often interested only in the potential differences between points, but in many circumstances it is customary to define the potential to be *zero* at an *infinite distance* from any charges of interest. We can then say what "the" potential is at some nearby point. Potential and potential difference are scalar quantities.

Suppose we have an electric field E in the positive x-direction (towards the right). This means that potential is decreasing to the right. You would have to do work to move a positive test charge Q to the left, so that potential is increasing towards the left. The force on Q is QE, so the work you would have to do to move it a distance dx to the right is $-QE\ dx$, but by definition this is also equal to $Q\ dV$, where dV is the potential difference between x and $x + dx$.

Therefore

$$E = -\frac{dV}{dx}. \qquad 2.1.1$$

In a more general three-dimensional situation, this is written

$$\mathbf{E} = -\mathbf{grad}V = -\nabla V = -\left(\mathbf{i}\frac{\partial V}{\partial x} + \mathbf{j}\frac{\partial V}{\partial x} + \mathbf{k}\frac{\partial V}{\partial x}\right). \qquad 2.1.2$$

We see that, as an alternative to expressing electric field strength in newtons per coulomb, we can equally well express it in volts per metre ($\mathrm{V\ m^{-1}}$).

The inverse of equation 2.1.1 is, of course,

$$V = -\int E\,dx + \text{constant}. \qquad 2.1.3$$

2.2 *Potential Near Various Charged Bodies*

2.2.1 *Point Charge*

Let us arbitrarily assign the value zero to the potential at an infinite distance from a point charge Q. "The" potential at a distance r from this charge is then the work required to move a unit positive charge from infinity to a distance r.

At a distance x from the charge, the field strength is $\dfrac{Q}{4\pi\epsilon_0 x^2}$. The work required to move a unit charge from x to $x + \delta x$ is $-\dfrac{Q\,\delta x}{4\pi\epsilon_0 x^2}$. The work required to move unit charge from r to infinity is $-\dfrac{Q}{4\pi\epsilon_0}\displaystyle\int_r^\infty \frac{dx}{x^2} = -\frac{Q}{4\pi\epsilon_0 r}$. The work required to move unit charge from infinity to r is minus this.

Therefore
$$V = +\frac{Q}{4\pi\epsilon_0 r}. \qquad 2.2.1$$

The *mutual potential energy* of two charges Q_1 and Q_2 separated by a distance r is the work required to bring them to this distance apart from an original infinite separation. This is

$$\text{P.E.} = +\frac{Q_1 Q_2}{4\pi\epsilon_0 r}. \qquad 2.2.2$$

Before proceeding, a little review is in order.

Field at a distance r from a charge Q:

$$E = \frac{Q}{4\pi\epsilon_0 r^2}, \qquad \text{N C}^{-1} \text{ or V m}^{-1}$$

or, in vector form,
$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} = \frac{Q}{4\pi\epsilon_0 r^3}\mathbf{r}. \qquad \text{N C}^{-1} \text{ or V m}^{-1}$$

Force between two charges, Q_1 and Q_2:

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon r^2}. \qquad \text{N}$$

Potential at a distance r from a charge Q:

$$V = \frac{Q}{4\pi\varepsilon_0 r}. \qquad \text{V}$$

Mutual potential energy between two charges:

$$\text{P.E.} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r}. \qquad \text{J}$$

We couldn't possibly go wrong with any of these, could we?

2.2.2 *Spherical Charge Distributions*

Outside any spherically-symmetric charge distribution, the field is the same as if all the charge were concentrated at a point in the centre, and so, then, is the potential. Thus

$$V = \frac{Q}{4\pi\varepsilon_0 r}. \qquad 2.2.3$$

Inside a *hollow spherical shell* of radius a and carrying a charge Q the field is zero, and therefore the potential is uniform throughout the interior, and equal to the potential on the surface, which is

$$V = \frac{Q}{4\pi\varepsilon_0 a}. \qquad 2.2.4$$

A solid sphere of radius a bearing a charge Q that is uniformly distributed throughout the sphere is easier to imagine than to achieve in practice, but, for all we know, a proton might be like this (it might be – but it isn't!), so let's calculate the field at a point P inside the sphere at a distance r ($< a$) from the centre. See figure II.1

We can do this in two parts. First the potential from the part of the sphere "below" P. If the charge is uniformly distributed throughout the sphere, this is just $\frac{Q_r}{4\pi\varepsilon_0 r}$. Here Q_r is the charge contained within radius r, which, if the charge is uniformly distributed throughout the sphere, is $Q(r^3/a^3)$. Thus, that part of the potential is $\frac{Qr^2}{4\pi\epsilon_0 a^3}\cdot$

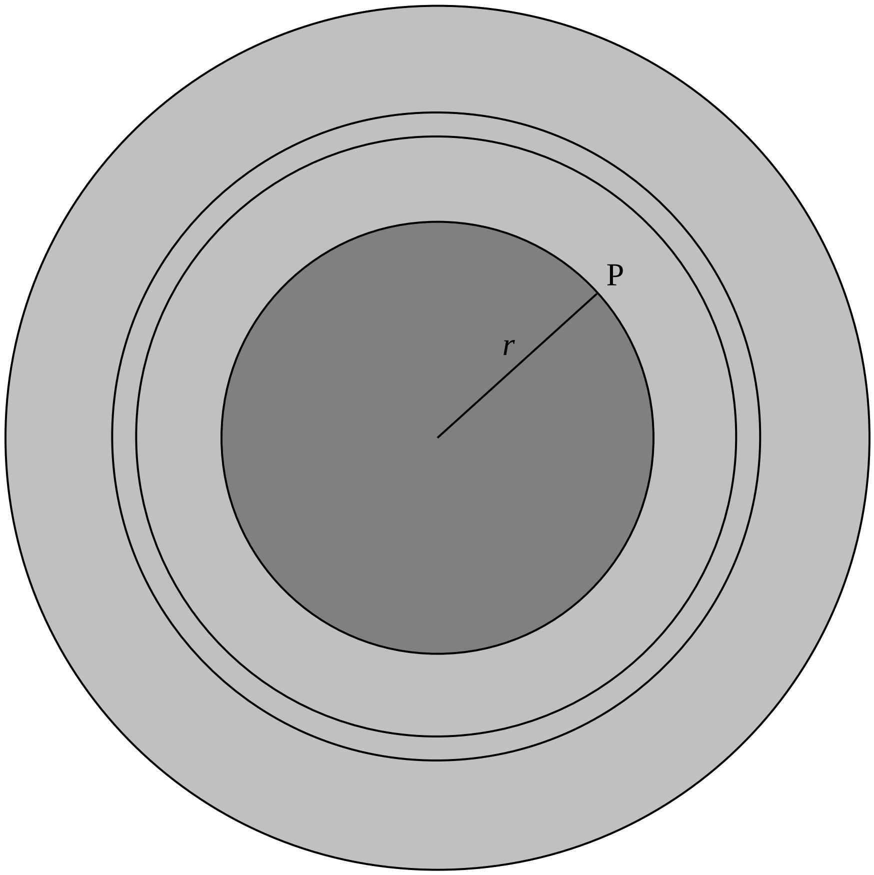


FIGURE II.1

Next, we calculate the contribution to the potential from the charge "above" P. Consider an elemental shell of radii x , $x + \delta x$. The charge held by it is $\delta Q = \frac{4\pi x^2 \delta x}{\frac{4}{3}\pi a^3}\times Q = \frac{3Qx^2\delta x}{a^3}\cdot$ The contribution to the potential at P from the charge in this elemental shell is $\frac{\delta Q}{4\pi\epsilon_0 x} = \frac{3Qx\delta x}{4\pi\epsilon_0 a^3}\cdot$ The contribution to the potential from all the charge "above" P is $\frac{3Q}{4\pi\epsilon_0 a^3}\int_r^a x\,dx = \frac{3Q(a^2-r^2)}{4\pi\epsilon_0 .2a^3}\cdot$ Adding together the two parts of the potential, we obtain

$$V = \frac{Q}{8\pi\epsilon_0 a^3}(3a^2 - r^2). \qquad 2.2.5$$

2.2.3 *Long Charged Rod*

The field at a distance r from a long charged rod carrying a charge λ coulombs per metre is $\frac{\lambda}{2\pi\epsilon_0 r}$. Therefore the potential difference between two points at distances a and b from the rod $(a < b)$ is

$$V_b - V_a = -\frac{\lambda}{2\pi\epsilon_0}\int_a^b \frac{dr}{r}.$$

□

$$V_a - V_b = \frac{\lambda}{2\pi\epsilon_0}\ln(b/a). \qquad 2.2.6$$

2.2.4 *Large Plane Charged Sheet*

The field at a distance r from a large charged sheet carrying a charge σ coulombs per square metre is $\frac{\sigma}{2\epsilon_0}$. Therefore the potential difference between two points at distances a and b from the sheet $(a < b)$ is

$$V_a - V_b = \frac{\sigma}{2\epsilon_0}(b - a). \qquad 2.2.7$$

2.2.5 *Potential on the Axis of a Charged Ring*

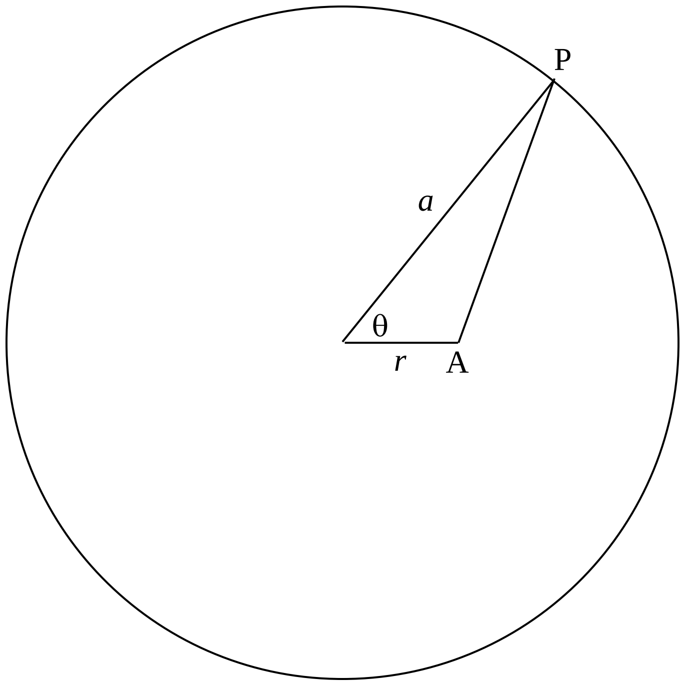
The field on the axis of a charged ring is given in section 1.6.4. The reader is invited to show that the potential on the axis of the ring is

$$V = \frac{Q}{4\pi\epsilon_0 (a^2 + x^2)^{1/2}}. \qquad 2.2.8$$

You can do this either by integrating the expression for the field or just by thinking about it for a few seconds and realizing that potential is a scalar quantity.

2.2.6 *Potential in the Plane of a Charged Ring*

We suppose that we have a ring of radius a bearing a charge Q. We shall try to find the potential at a point in the plane of the ring and at a distance r $(0 \le r < a)$ from the centre of the ring.



Consider an element $\delta\theta$ of the ring at P. The charge on it is $\dfrac{Q\delta\theta}{2\pi}$. The potential at A due this element of charge is

$$\frac{1}{4\pi\epsilon_0}\cdot\frac{Q\delta\theta}{2\pi}\cdot\frac{1}{\sqrt{a^2 + r^2 - 2ar\cos\theta}} = \frac{Q}{4\pi\epsilon_0 . 2\pi a}\cdot\frac{\delta\theta}{\sqrt{b - c\cos\theta}}\,, \qquad 2.2.9$$

where $b = 1 + r^2/a^2$ and $c = 2r/a$. The potential due to the charge on the entire ring is

$$V = \frac{Q}{4\pi\epsilon_0 . \pi a}\int_0^\pi \frac{d\theta}{\sqrt{b - c\cos\theta}}\,. \qquad 2.2.10$$

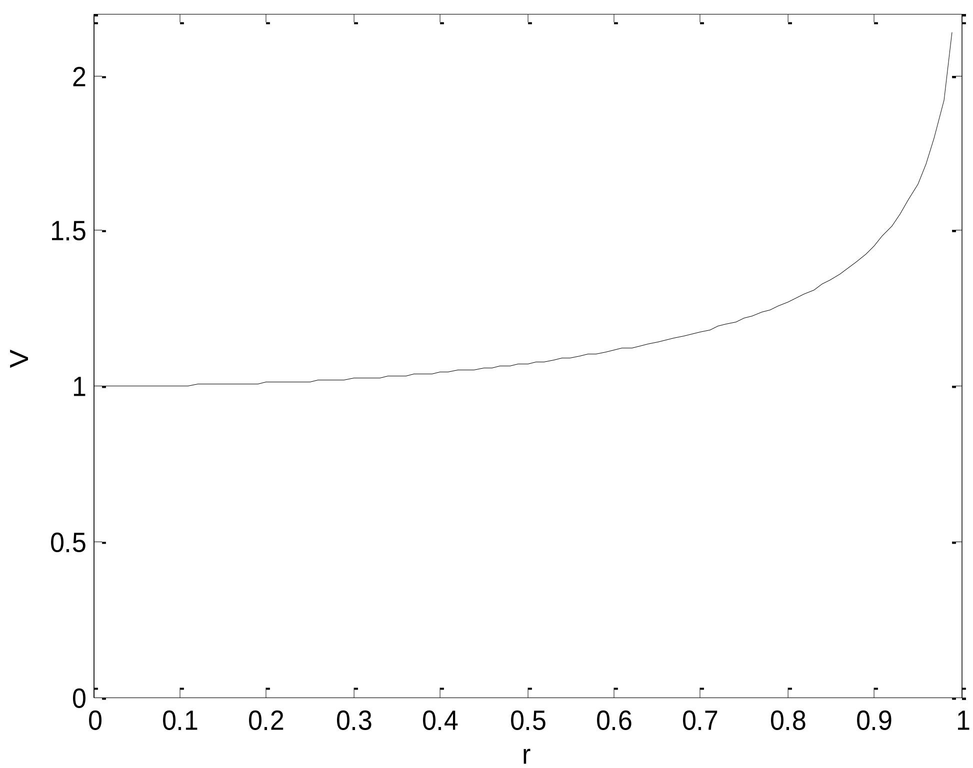
This requires a numerical integration for each value of r/a. For those familiar with elliptic integrals, this can also be written:

$$V = \frac{Q}{4\pi\epsilon_0 a} \cdot \frac{2}{\pi} \cdot \frac{K(k^2)}{(1+x)} \ , \qquad 2.2.11$$

where $K(k^2)$ is the elliptic integral of the first kind, and $k^2 = \dfrac{4r/a}{1+(r/a)^2}$. Thus the numerical integration can be avoided if one has available a table of elliptic integrals. Most of us are not familiar with the function $K(k^2)$, nor do we have a table of the function, so we cannot avoid a numerical integration, and we have to evaluate the elliptic integral by numerical integration of

$$K(k^2) = \int_0^{\pi/2} \frac{d\psi}{\sqrt{1-k^2 \sin^2 \psi}} \ . \qquad 2.2.12$$

I have evaluated the potential by numerical integration of equation 2.2.10, and also by numerical integration of equations 2.2.11 and 2.2.12, with nine-digit agreement between the two methods, with the result shown below.



The field is equal to the gradient of this and is directed towards the centre of the ring. It looks as though a small positive charge would be in stable equilibrium at the centre of the ring, and this would be so if the charge were constrained to remain in the plane of the ring. But, without such a constraint, the charge would be pushed away from the ring if it strayed at all above or below the plane of the ring. In other words, in three dimensions, the potential at the centre is a saddle-point.

2.2.7 *Potential on the Axis of a Charged Disc*

The field on the axis of a charged disc is given in section 1.6.5. The reader is invited to show that the potential on the axis of the disc is

$$V = \frac{2Q}{4\pi\epsilon_0 a^2}[(a^2 + x^2)^{1/2} - x]. \qquad 2.2.9$$

2.3 *Electron-volts*

The *electron-volt* is a unit of *energy* or *work*. An electron-volt (eV) is the work required to move an electron through a potential difference of one volt. Alternatively, an electron-volt is equal to the kinetic energy acquired by an electron when it is accelerated through a potential difference of one volt. Since the magnitude of the charge of an electron is about 1.602×10^{-19} C, it follows that an electron-volt is about 1.602×10^{-19} J. Note also that, because the charge on an electron is negative, it requires work to move an electron from a point of high potential to a point of low potential.

Exercise. If an electron is accelerated through a potential difference of a million volts, its kinetic energy is, of course, 1 MeV. At what speed is it then moving?

First attempt. $\frac{1}{2}m\upsilon^2 = eV.$

(Here *eV*, written in italics, is not intended to mean the unit electron-volt, but *e* is the magnitude of the electron charge, and *V* is the potential difference (10^6 volts) through which it is accelerated.) Thus $\upsilon = \sqrt{2eV/m}$. With $m = 9.109 \times 10^{-31}$ kg, this comes to $\upsilon = 5.9 \times 10^8$ m s^{-1}. Oops! That looks awfully fast! We'd better do it properly this time.

Second attempt. $(\gamma - 1)mc^2 = eV.$

Some readers will know exactly what we are doing here, without explanation. Others may be completely mystified. For the latter, the difficulty is that the speed that we had calculated was even greater than the speed of light. To do this properly we have to use the formulas of special relativity. See, for example, Chapter 15 of the Classical Mechanics section of these notes.

At any rate, this results in $\gamma = 2.958$, whence $\beta = 0.9411$ and $\upsilon = 2.82 \times 10^8$ m s^{-1}.

2.4 *A Point Charge and an Infinite Conducting Plane*

An infinite plane metal plate is in the *xy*-plane. A point charge $+Q$ is placed on the z-axis at a height h above the plate. Consequently, electrons will be attracted to the part of the plate immediately below the charge, so that the plate will carry a negative charge density σ which is greatest at the origin and which falls off with distance ρ from the origin. Can we determine $\sigma(\rho)$? See figure II.2

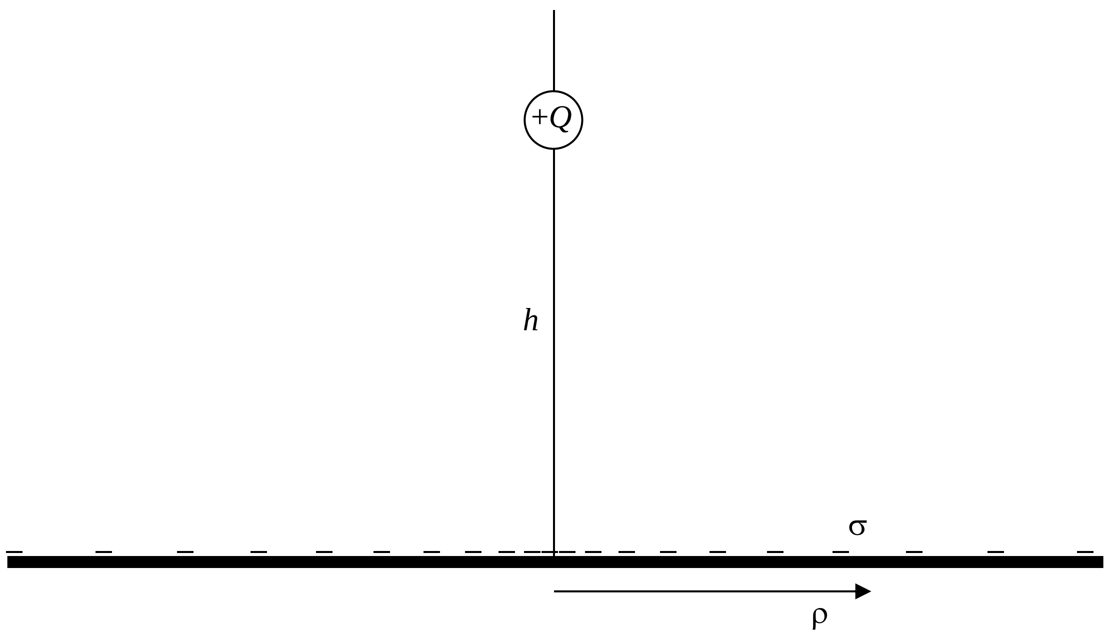


FIGURE II.2

First, note that the metal surface, being a conductor, is an *equipotential* surface, as is any metal surface. The potential is uniform anywhere on the surface. Now suppose that, instead of the metal surface, we had (in addition to the charge $+Q$ at a height h above the *xy*-plane), a second point charge, $-Q$, at a distance h below the *xy*-plane. The potential in the *xy*-plane would, by symmetry, be uniform everywhere. That is to say that the potential in the *xy*-plane is the same as it was in the case of the single point charge and the metal plate, and indeed the potential at any point above the plane is the same in both cases. For the purpose of calculating the potential, we can replace the metal plate by an *image* of the point charge. It is easy to calculate the potential at a point $(z\ ,\ \rho)$. If we suppose that the permittivity above the plate is ε_0, the potential at $(z\ ,\ \rho)$ is

$$V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{[\rho^2 + (h-z)^2]^{1/2}} - \frac{1}{[\rho^2 + (h+z)^2]^{1/2}}\right). \qquad 2.4.1$$

The field strength E in the xy-plane is $-\partial V/\partial z$ evaluated at $z = 0$, and this is

$$E = -\frac{2Q}{4\pi\epsilon_0}\cdot\frac{h}{(\rho^2 + h^2)^{3/2}} \cdot \qquad 2.4.2$$

The D-field is ϵ_0 times this, and since all the lines of force are above the metal plate, Gauss's theorem provides that the charge density is $\sigma = D$, and hence the charge density is

$$\sigma = -\frac{Q}{2\pi}\cdot\frac{h}{(\rho^2 + h^2)^{3/2}} \cdot \qquad 2.4.3$$

This can also be written $\qquad \sigma = -\frac{Q}{2\pi}\cdot\frac{h}{\xi^3}\ , \qquad 2.4.4$

where $\xi^2 = \rho^2 + h^2$, with obvious geometric interpretation.

Exercise: How much charge is there on the surface of the plate within an annulus bounded by radii ρ and $\rho + d\rho$? Integrate this from zero to infinity to show that the total charge induced on the plate is $-Q$.

2.5 *A Point Charge outside a Conducting Sphere*

I am indebted here to Alain Charbonneau who drew my attention to the distinction between the two cases – where the sphere is *grounded* (in UK nomenclature "earthed"):

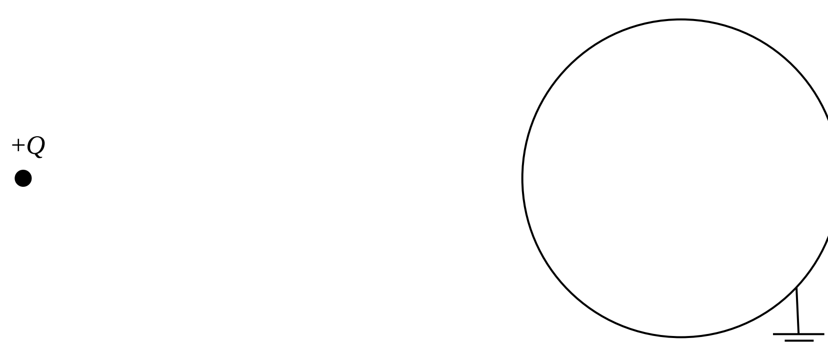


FIGURE II.3a

and where it is *isolated*:

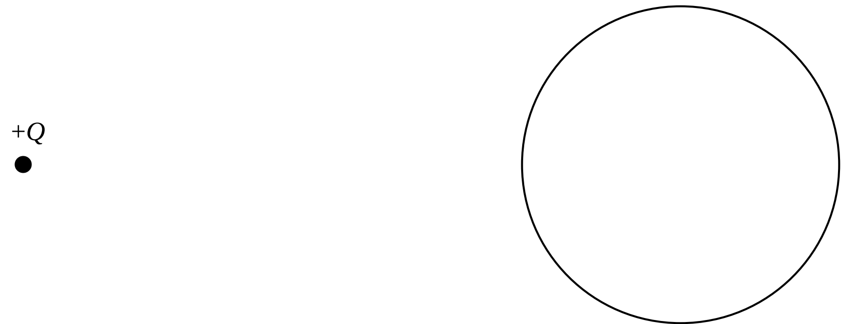


FIGURE II.3b

and for describing the physics of the differences between the two cases with exceptional lucidity and limpidity.

Before embarking on the calculation it may be useful to imagine qualitatively what happens in the two cases.

<u>Total charge on the sphere</u>

We suppose that intially the sphere is uncharged.

If the sphere is <u>isolated</u>, the total charge on it remains zero, regardless of the distance of the point charge.

If the sphere is <u>grounded</u>, it acquires a negative charge (from the ground), which increases as it is approached by the positive point charge. This charge is distributed (not uniformly) over the surface of the sphere.

<u>Potential of the sphere</u>

In both cases – isolated and grounded – the <u>potential is uniform over the surface</u> of the conducting sphere. If this were not so, a current would flow over the surface until the potential were to become uniform.

If the sphere is grounded, its potential is the same as the potential of the ground, taken to be zero.

If the sphere is isolated, its potential is positive , owing to the presence of the point charge. If it is then grounded, it sucks electrons from the ground until its potential drops to zero.

Electric field inside the sphere

Since the potential in both cases (isolated and grounded) is uniform over the surface of the sphere, the electric field **E** inside the sphere is zero for both cases (isolated and grounded) regardless of the distance of the point charge.

Distribution of surface charge density over the surface of the sphere

If the sphere is isolated, the surface charge density is not uniform. A portion of the sphere facing the external point charge has a negative surface charge density; the remainder of the sphere has a positive surface charge density. The total charge on the sphere is zero. The electric field at a point inside the sphere is the vector sum of the field from the external point charge and the charge distribution over the surface of the sphere, and, as pointed out in the previous paragraph, this field is zero.

If the sphere is now grounded, it acquires a negative charge. Since the field inside the sphere is still zero, this newly-acquired electric charge must be distributed uniformly over the surface. The net surface charge density at a point is the sum of the non-uniformly distributed surface charge density arising from the presence of the external point charge and the uniformly-distributed negative charge sucked from the ground.

Having understood the problem qualitatively, we are now ready to embark upon the quantitative calculation.

2.5a *The sphere is grounded*

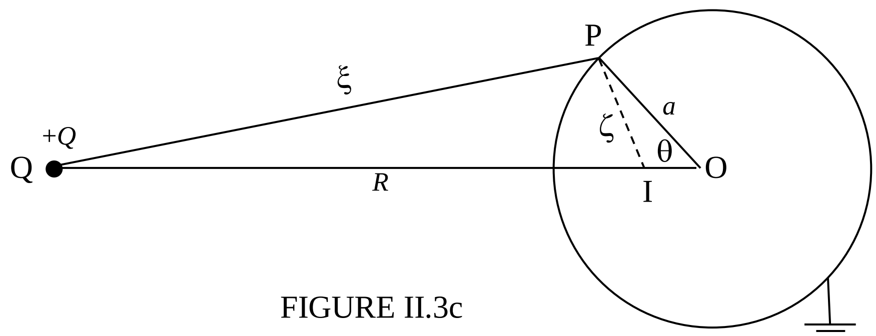


FIGURE II.3c

A point charge $+Q$ is at a point Q at a distance R from the centre O of a grounded metal sphere of radius a. We are going to try to calculate the surface charge density induced on the surface of the sphere, as well as the total charge induced on the sphere, as a function of position on the surface.

Let us construct a point I such that the triangles OPI and OQP are similar, with the lengths shown in figure II.3c. The length OI is a^2/R. Then $R/\xi = a/\zeta$, or

$$\frac{1}{\xi} - \frac{a/R}{\zeta} = 0. \qquad 2.5.1$$

This relation between the variables ξ and ζ is in effect the equation to the sphere expressed in these variables.

Now suppose that, instead of the metal sphere, we had (in addition to the charge $+Q$ at a distance R from O), a second point charge $-(a/R)Q$ at I. The locus of points where the *potential is zero* is where

$$\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{\xi} - \frac{a/R}{\zeta}\right) = 0. \qquad 2.5.2$$

That is, the surface of our sphere. Thus, for purposes of calculating the potential, we can replace the metal sphere by an *image* of Q at I, this image carrying a charge of $-(a/R)Q$.

Let us take the line OQ as the z-axis of a coordinate system. Let X be some point such that OX = r and the angle XOQ = θ. The potential at P from a charge $+Q$ at Q and a charge $-(a/R)Q$ at I is (see figure II.3d)

$$V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{(r^2+R^2-2rR\cos\theta)^{1/2}} - \frac{a/R}{(r^2+a^4/R^2-2a^2r\cos\theta/R)^{1/2}}\right) = 0. \qquad 2.5.2$$

The E field on the surface of the sphere is $-\partial V/\partial r$ evaluated at $r = a$. The D field is ε_0 times this, and the surface charge density is equal to D. After some patience and algebra, we obtain, for a point on the surface of the sphere (where, of course, X and P are coincident),

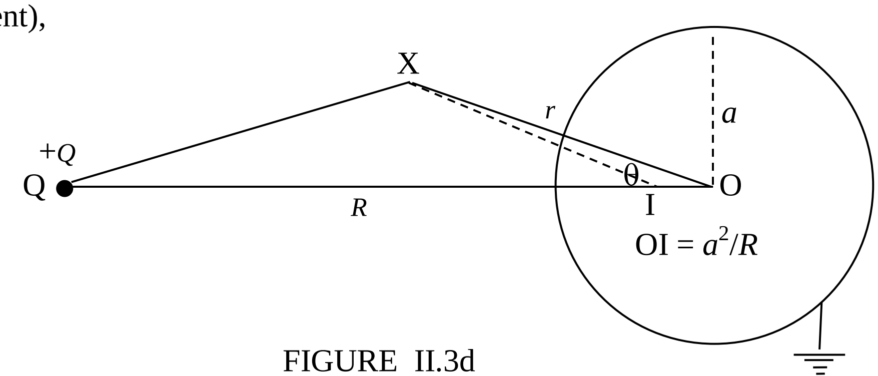


FIGURE II.3d

$$\sigma = -\frac{Q}{4\pi}\cdot\frac{R^2-a^2}{a}\cdot\frac{1}{\left(a^2+R^2-2aR\cos\theta\right)^{3/2}} = \frac{-Q\left(R^2-a^2\right)}{4\pi a\xi^3}$$

2.5.3

Let $\rho = \dfrac{R}{a}$ (dimensionless); then equation 2.5.3 can be written

$$\sigma = -\frac{Q(\rho^2-1)}{4\pi a^2}\cdot\frac{1}{(1+\rho^2-2\rho\cos\theta)^{3/2}}$$

2.5.4

This is shown in figure 11.4a for for $\rho =$ 2, 3.5 and 5, in which the surface charge density is in units of $-Q/(4\pi a^2)$.

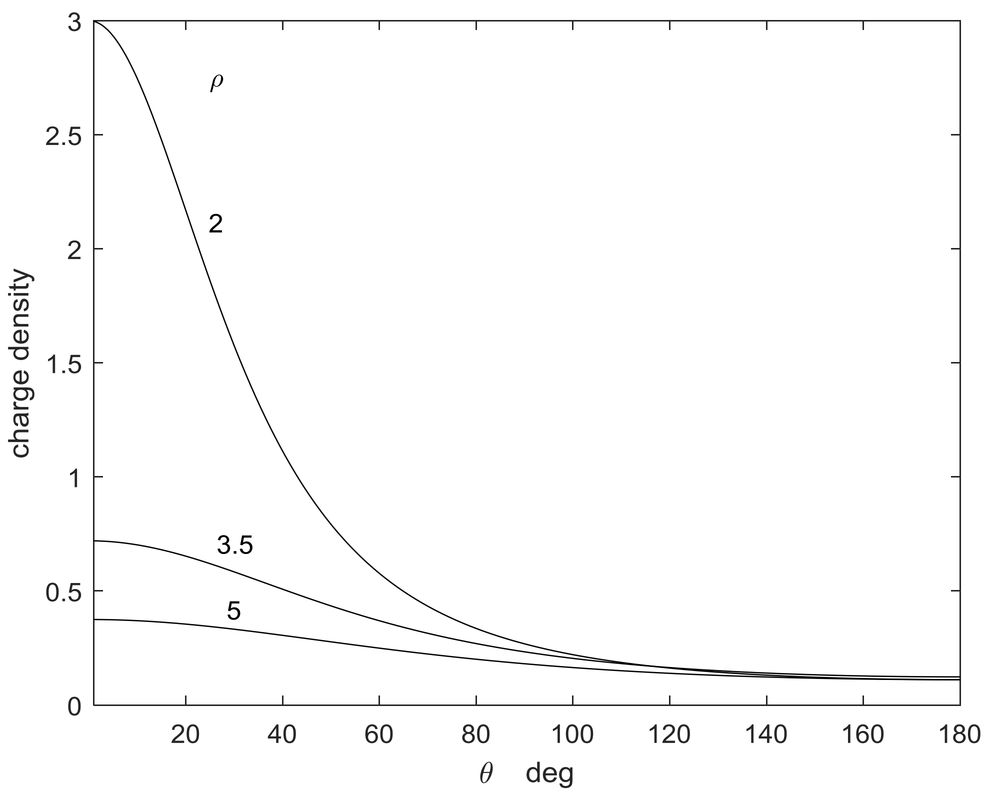


FIGURE 11.4a

It may be of interest to see how the surface charge density on the sphere at $\theta = 0$ varies with ρ, the distance of the point charge from the centre of the sphere. A quick uncritical look at equation 2.5.4 might give the quite wrong impression that when $\rho = 1$, $\sigma = 0$, but this, of course, is quite wrong! For $\theta = 0$, equation 2.5.4 becomes

$$\sigma(\theta = 0) \;=\; \frac{\rho + 1}{(\rho - 1)^2}$$

2.5.5

where again σ is in units of $-Q/(4\pi a^2)$.

That looks like this:

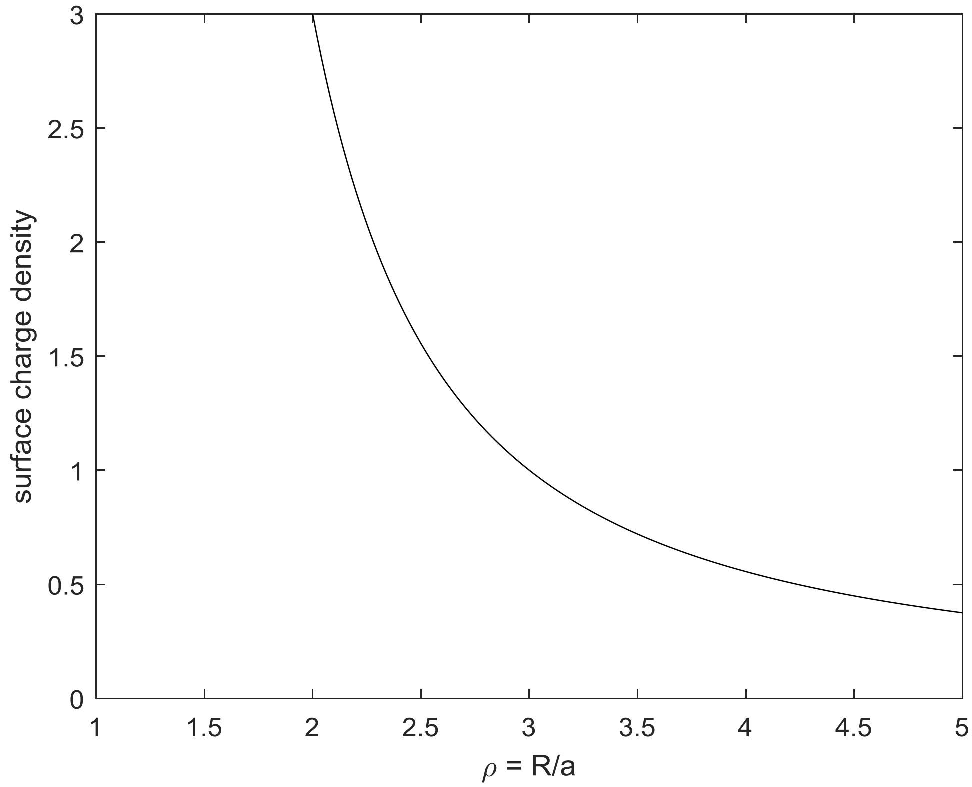


FIGURE 11.4b

As $\rho \to 1$, i.e. as Q approaches the surface of the sphere, the surface charge density at θ = 0 approaches infinity. That indeed would be so if there were truly such a thing as a mathematical point charge, for which Coulomb's law avers that the electric field at the origin is infinite. A proton is a very small charge-bearing object. If you assume it is a sphere and if you can find its radius in a book somewhere, it might be of interest to calculate the electric field at the surface of a proton. What about an electron? Is it a mathematical point? You'd have to ask a particle physicist.

What is the total charge induced on the sphere?

The area of an elemental zone of the sphere between θ and $\theta + d\theta$ is $2\pi a^2 \sin\theta d\theta$, and so the charge on such a zone is

$$-\tfrac{1}{2}Q(\rho^2 - 1) \cdot \frac{\sin\theta\, d\theta}{(1 + \rho^2 - 2\rho\cos\theta)^{3/2}}$$

2.5.6

The total charge on the sphere is then the integral of this from θ = 0 to π, which is

$$-Q/\rho \ .$$

2.5.7

That amount of electronic charge has been dragged out of the ground (earth). We have argued that this charge must be uniformly distributed over the surface, thus contributing

$$-\frac{Q}{4\pi a^2 \rho} \qquad 2.5.8$$

to the overall surface charge density.

2.5b *The sphere is isolated*

When the sphere is grounded, its surface charge density is given by equation 2.5.4. To find the surface charge density when the sphere is isolated, we must subtract the expression 2.5.8 . Thus the expression for the surface charge density when the sphere is isolated is

$$\sigma = -\frac{Q}{4\pi a^2}\left[\frac{\rho^2 - 1}{(1 + \rho^2 - 2\rho\cos\theta)^{3/2}} - \frac{1}{\rho}\right]$$

2.5.9

For ρ = 2, this looks like:

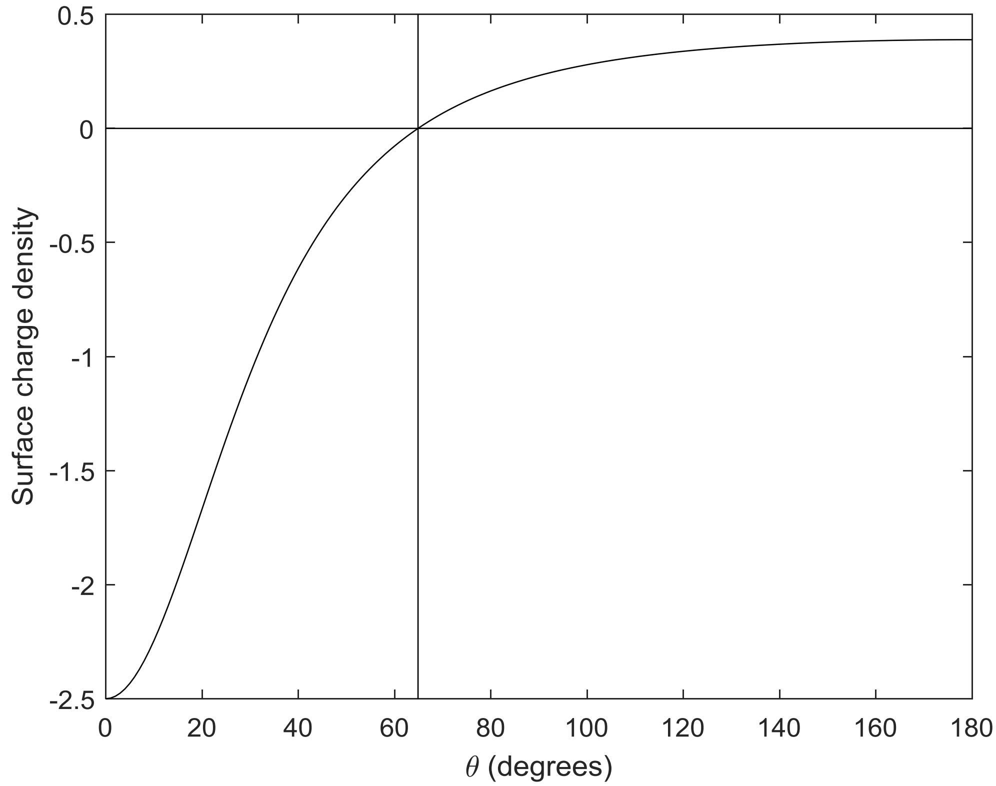


FIGURE 11.4c

in which σ is in units of $Q/(4\pi a^2)$.

To find the total charge on the isolated sphere we must, as before, mutiply this by the area of an elemental zone on the sphere, namely $2\pi a^2 \sin\theta d\theta$, and integrate 0 to π. This comes, of course, to zero.

The surface charge density is negative on a portion of the sphere facing the external point charge, and positive on the remainder of the sphere. It changes sign (i.e. it is zero) where the square bracket in equatiom 2.5.9 is zero. That is, where

$$\cos\theta = \frac{1 + \rho^2 - (\rho^3 - \rho)^{2/3}}{2\rho} \tag{2.5.10}$$

That is shown in figure II.4d. The reader should ask him/herself if that makes physical sense and is about what one would expect.

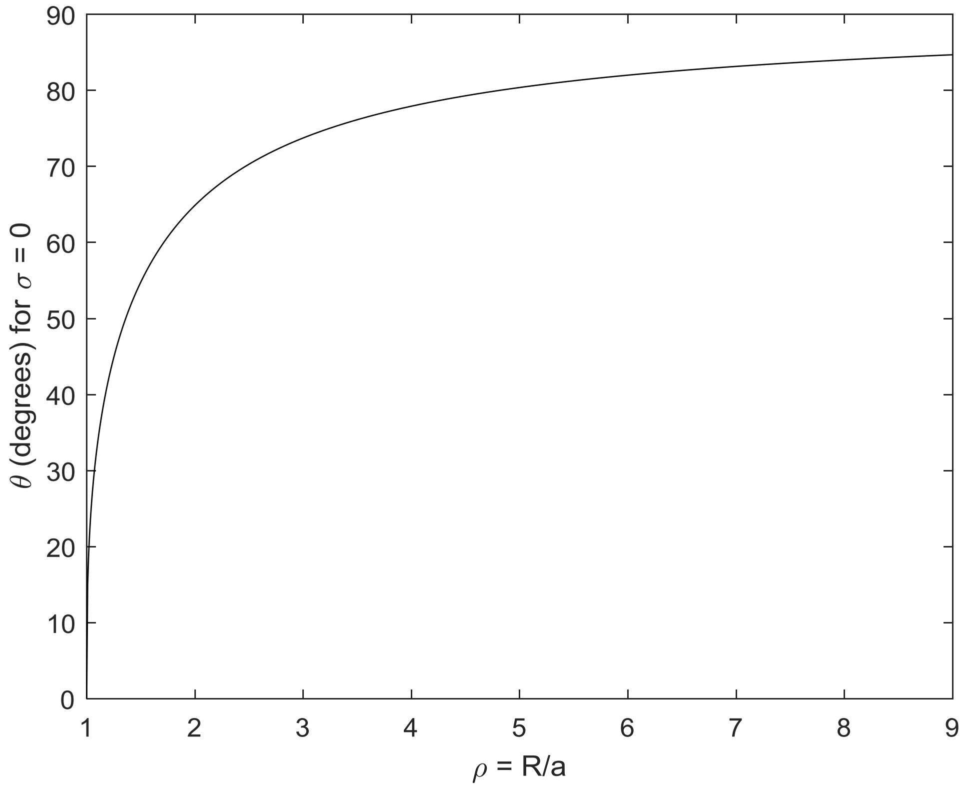


FIGURE 11.4d

The sphere has developed a dipole moment, and it will be of interest to calculate the induced dipole moment as a funtion of ρ, the distance of the external point charge. The subject of dipole moments is dealt with in Chapter 3, and the reader may wish to read Chapter 3 before proceeding with the calculation in this instance.

Since the isolated sphere carries no net charge, the dipole moment is independent of the position of the origin of coordinates. It will be convenient to take the origin to be at the centre of the sphere.

There are two methods of calculating the dipole moment. The first is the simple and straightforward method shown to me by Alain Charbonneau. The second is the much more difficult and clumsy method initially adopted by myself. It is gratifying that both give the same result!

The surface charge density over the surface of he isolated sphere varies with position over the surface of the sphere as given by equation 2.5.7, and as shown in figure II.4d for $\rho = 2$, although the total charge on the isolated sphere is zero. We have seen that the electric field and potential outside the sphere can be calculated as if there were a point image charge $-Q/\rho$ at a distance a/ρ from the centre of the sphere, plus a charge $+Q/\rho$ distributed uniformly over the the surface. That is, it can be calculated as if there were a point image charge $-Q/\rho$ at a distance a/ρ from the centre of the sphere, plus a point charge $+Q/\rho$ at the centre of the sphere. Thus we immediately see that the induced dipole moment of the isolated sphere is Qa/ρ^2.

The clumsy effort that I initially used was to note that the surface charge density at angle θ is given by equation 2.5.9. The area of an elemental zone between θ and $\theta + d\theta$ is $2\pi a^2 \sin\theta \, d\theta$, so the total charge on such a zone is the product of these two expressions, and the moment of the charge on this zone is this product times the distance $a \cos\theta$ of the zone from centre. Integration of this long expression from $\theta = 0$ to 180° gratifyingly gives the same result Qa/ρ^2 as before.

2.6 *Two Semicylindrical Electrodes*

This section requires that the reader should be familiar with functions of a complex variable and conformal transformations. For readers not familiar with these, this section can be skipped without prejudice to understanding following chapters. For readers who are familiar, this is a nice example of conformal transformations to solve a physical problem.

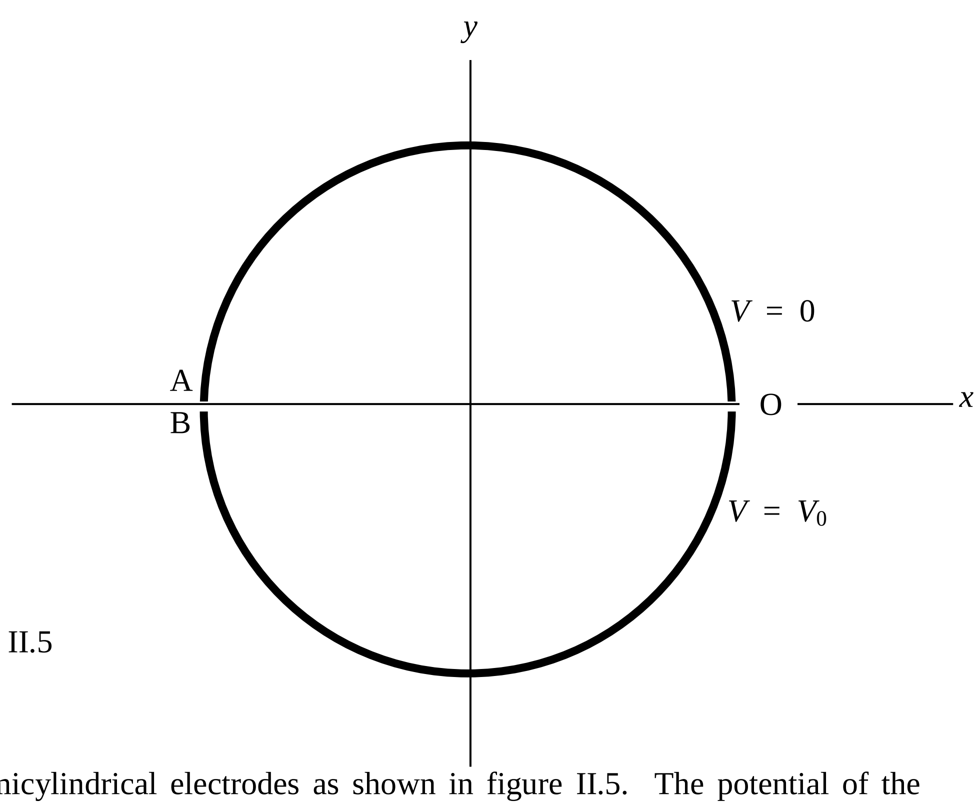


FIGURE II.5

We have two semicylindrical electrodes as shown in figure II.5. The potential of the upper one is 0 and the potential of the lower one is V_0. We'll suppose the radius of the curcle is 1; or, what amounts to the same thing, we'll express coordinates x and y in units of the radius. Let us represent the position of any point whose coordinates are $(x\ ,\ y)$ by a complex number $z = x + iy$.

Now let $w = u + i\upsilon$ be a complex number related to z by $w = i\left(\dfrac{1-z}{1+z}\right)$; that is, $z = \dfrac{1+iw}{1-iw}$. Substitute $w = u + i\upsilon$ and $z = x + iy$ in each of these equations, and equate real and imaginary parts, to obtain

$$u = \frac{2y}{(1+x)^2 + y^2}; \qquad \upsilon = \frac{1-x^2-y^2}{(1+x)^2+y^2}; \tag{2.6.1}$$

$$x = \frac{1-u^2-\upsilon^2}{u^2+(1+\upsilon)^2}; \qquad y = \frac{2u}{u^2+(1+\upsilon)^2}. \tag{2.6.2}$$

In that case, the upper semicircle ($V = 0$) in the xy-plane maps on to the positive u-axis in the $u\upsilon$-plane, and the lower semicircle ($V = V_0$) in the xy-plane maps on to the negative u-axis in the $u\upsilon$-plane. (Figure II.6.) Points inside the circle bounded by the electrodes in the xy-plane map on to points above the u-axis in the $u\upsilon$-plane.

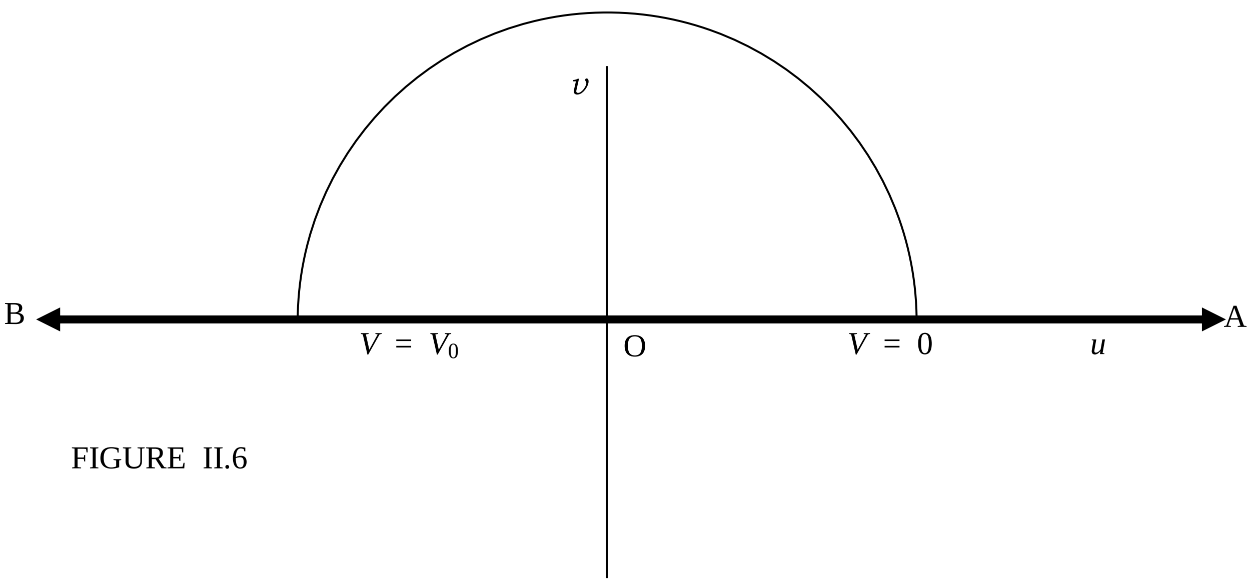


FIGURE II.6

In the $u\upsilon$-plane, the lines of force are semicircles, such as the one shown. The potential goes from 0 at one end of the semicircle to V_0 at the other, and so equation to the semicircular line of force is

$$\frac{V}{V_0} = \frac{\arg w}{\pi} \tag{2.6.3}$$

or
$$V = \frac{V_0}{\pi}\tan^{-1}(\upsilon / u). \tag{2.6.4}$$

The equipotentials (V = constant) are straight lines in the $u\upsilon$-plane of the form

$$\upsilon = fu. \tag{2.6.5}$$

(You would prefer me to use the symbol m for the slope of the equipotentials, but in a moment you will be glad that I chose the symbol f.)

If we now transform back to the xy-plane, we see that the equation to the lines of force is

$$V = \frac{V_0}{\pi}\tan^{-1}\left(\frac{1-x^2-y^2}{2y}\right), \tag{2.6.6}$$

and the equation to the equipotentials is

$$1 - x^2 - y^2 = 2fy, \tag{2.6.7}$$

or
$$x^2 + y^2 + 2fy - 1 = 0. \tag{2.6.8}$$

Now aren't you glad that I chose f ? Those who are handy with conic sections (see Chapter 2 of Celestial Mechanics) will understand that the equipotentials in the xy-plane are circles of radii $\sqrt{f^2+1}$, whose centres are at $(0, \pm f)$, and which all pass through the points $(\pm 1, 0)$. They are drawn as blue lines in figure II.7. The lines of force are the orthogonal trajectories to these, and are of the form

$$x^2 + y^2 + 2gy + 1 = 0. \tag{2.6.9}$$

These are circles of radii $\sqrt{g^2 - 1}$ and have their centres at $(0, \pm g)$. They are shown as dashed red lines in figure II.7.

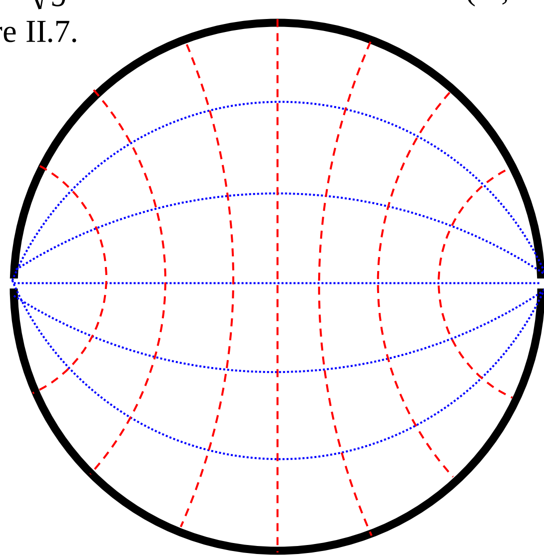


FIGURE II.7