CHAPTER 9
CONSERVATIVE FORCES

9.1 *Introduction.*

In Chapter 7 we dealt with forces on a particle that depend on the speed of the particle. In Chapter 8 we dealt with forces that depend on the time. In this chapter, we deal with forces that depend only on the position of a particle. Such forces are called *conservative forces*. While only conservative forces act, the sum of potential and kinetic energies is conserved.

Conservative forces have a number of properties. One is that the work done by a conservative force (or, what amounts to the same thing, the *line integral* of a conservative force) as it moves from one point to another is route-independent. The work done depends only on the coordinates of the beginning and end points, and not on the path taken to get from one to the other. It follows from this that the work done by a conservative force, or its line integral, round a closed path is zero. (If you are reminded here of the properties of a *function of state* in thermodynamics, all to the good.) Another property of a conservative force is that it can be derived from a potential energy function. Thus for any conservative force, there exists a scalar function $V(x, y, z)$ such that the force is equal to $-$**grad** V, or $-\nabla V$. In a one-dimensional situation, a sufficient condition for a force to be conservative is that it is a function of its position alone. In two- and three-dimensional situations, this is a necessary condition, but it is not a sufficient one. That a conservative force must be derivable from the gradient of a potential energy function and that its line integral around a closed path must be zero implies that the **curl** of a conservative force must be zero, and indeed a zero **curl** is a necessary and a sufficient condition for a force to be conservative.

This is all very well, but suppose you are stuck in the middle of an exam and your mind goes blank and you can't think what a line integral or a **grad** or a **curl** are, or you never did understand them in the first place, how can you tell if a force is conservative or not? Here is a rule of thumb that will almost never fail you: If the force is the tension in a stretched elastic string or spring, or the thrust in a compressed spring, or if the force is gravity or if it is an electrostatic force, the force is conservative. If it is not one of these, it is not conservative.

Example. A man lifts up a basket of groceries from a table. Is the force that he exerts a conservative force?

Answer: No, it is not. The force is not the tension in a string or a spring, nor is it electrostatic. And, although he may be fighting against gravity, the force that he exerts with his muscles is not a gravitational force. Therefore it is not a conservative force. You see, he may be accelerating as he moves the basket up, in which case the force that he is exerting is greater than the weight of the groceries. If he is moving at constant speed, the force he exerts is equal to the weight of the groceries. Thus the force he exerts depends on whether he is accelerating or not; the force does not depend only on the position.

Example. But you are not in an exam now, and you have ample time to remind yourself what a **curl** is. Each of the following two forces are functions of position only - a necessary condition for them to be conservative. But it is not a sufficient condition. In fact one of them is conservative and the other isn't. You will have to find out by evaluating the **curl** of each. The one that has zero **curl** is the conservative one. When you have identified it, work out the potential energy function from which it can be derived. In other words, find $V(x,y,z)$ such that $\mathbf{F} = -\nabla V$.

i. $\mathbf{F} = (3x^2z - 3y^2z)\mathbf{i} - 6xyz\mathbf{j} + (x^3 - xy^2)\mathbf{k}$
ii. $\mathbf{F} = ax^2yz\mathbf{i} + bxy^2z\mathbf{j} + cxyz^2\mathbf{k}$

When you have identified which of these forces is irrotational (i.e. has zero **curl**), you can find the potential function by calculating the work done when the force moves from the origin to (x, y, z) along any route you choose. Indeed, you might try more than one route to convince yourself that the line integral is route-independent.

One could devise many exercises in determining whether various force functions are conservative, and, if so, what the corresponding potential energy functions are, but I am going to restrict this chapter to just one more topic, namely

9.2 *The Time and Energy Equation*

Consider a one-dimensional situation in which there is a force $F(x)$ that depends on the one coordinate only and is therefore a conservative force. If a particle moves under this force, its equation of motion is

$$m\ddot{x} = F(x) \tag{9.2.1}$$

and we can obtain the space integral in the usual fashion by writing $\ddot{x}$ as $v\dfrac{dv}{dx}\cdot$

Thus

$$mv\frac{dv}{dx} = F(x). \tag{9.2.2}$$

Integration yields

$$\tfrac{1}{2}mv^2 = \int F(x)dx + T_0. \tag{9.2.3}$$

Here $\frac{1}{2}mv^2$ is called the *kinetic energy* and the integration constant T_0 can be interpreted as the initial kinetic energy. Thus the gain in kinetic energy is

$$T - T_0 = \int F(x)dx, \tag{9.2.4}$$

the right hand side merely being the work done by the force.

Since F is a function of x alone, we can find a V such that $F = -dV/dx$. [It is true that we could also find a function V such that $F = +dV/dx$, but we shall shortly find that the choice of the minus sign gives V a desirable property that we can make use of.] If we integrate this equation, we find

$$V = -\int F(x)dx + V_0. \qquad 9.2.5$$

Here V is the *potential energy* and V_0 is the initial potential energy. From equations 9.2.4 and 9.2.5 we obtain

$$V + T = V_0 + T_0. \qquad 9.2.6$$

Thus the quantity $V + T$ is conserved under the action of a conservative force. (This would not have been the case if we had chosen the + sign in our definition of V.) We may call the sum of the two energies E, the total energy, and we have

$$T = E - V(x) \qquad 9.2.7$$

or

$$\tfrac{1}{2}m\upsilon^2 = E - V(x). \qquad 9.2.8$$

With $\upsilon = dx/dt$, we obtain, by integrating equation 9.2.8,

$$t = \pm\sqrt{\frac{m}{2}}\int_{x_0}^{x}\frac{dx}{\sqrt{E - V(x)}}. \qquad 9.2.9a$$

This may at first appear to be a very formal and laborious way of arriving at something very obvious and something we have known since we first studied physics, but we shall see that it can often be a quite useful equation. You might, by the way, check that this equation is dimensionally correct.

The choice of the sign in equation 9.2.9a may require some care, as will be evident in the examples that follow in the next section. If the particle is moving away from the origin, then its speed is $\upsilon = dx/dt$, and we choose the positive sign. If the particle is moving towards from the origin, then its speed is $\upsilon = -dx/dt$, and we choose the negative sign. However, I believe the following to be true: If the particle is moving away from the origin, then the initial value of x is smaller than the final value. If the particle is moving toward the origin, then the initial value of x is larger than the final value. It would seem to be safe, then, always to use the positive sign, but then the lower limit of integration is the smaller value of x (not necessarily the initial value), and the upper limit of integration is the larger value of x (not necessarily the final value). It may therefore be easier to write the equation in the form

$$t = +\sqrt{\frac{m}{2}}\int_{x_{\text{smaller}}}^{x_{\text{larger}}}\frac{dx}{\sqrt{E - V(x)}}. \qquad 9.2.9b$$

All that this means is that, for a conservative force, the time taken for a "return" journey is just equal to the time taken for the outbound journey, so one might as well always calculate the time for the outbound journey.

In some classes of problem such as pendulums, or rods falling over, the potential energy can be written as a function of an angle, and the kinetic energy is rotational kinetic energy written in the form $\frac{1}{2}I\omega^2$, where $\omega = d\theta / dt$. In that case, equation 9.2.9 takes the form

$$t = \pm\sqrt{\frac{I}{2}}\int_{\theta_0}^{\theta}\frac{d\theta}{\sqrt{E - V(\theta)}}. \qquad 9.2.10$$

You should check that this, too, is dimensionally correct.

9.3 *Examples*

(i) How long does it take a particle to fall to the ground from a height h, with a uniform acceleration g? From elementary methods you know that the answer is $\sqrt{2h/g}$, but we are going to use this simple example to illustrate the use of equation 9.2.9. The use of the equation to solve such a simple problem might seem a rather tedious way of solving an elementary problem, but the method is useful in more difficult cases, and the use of the method is best introduced with a simple example.

Let us take the origin to be the ground, and we shall measure distances y upward from the ground. The speed is then given by $v = -dy/dt$, so we use the negative sign in equation 9.2.9a. The initial and final values of y are h and 0 respectively. Alternatively, you can use equation 9.2.9b, with the positive sign, in which case the loser and upper limits of integration are 0 and h respectively.

The total energy E is the initial potential energy mgh (since the initial kinetic energy is zero), and the potential energy at height y is $V(y) = mgy$. Equation 9.2.9 therefore takes the form

$$t = -\sqrt{\frac{m}{2}}\int_{h}^{0}\frac{dy}{\sqrt{mgh - mgy}}. \qquad 9.3.1$$

This yields the expected answer $\sqrt{2h/g}$.

(ii) How long does it take for a particle, thrown vertically upwards with initial speed v_0, to reach a height h? Again, by elementary methods, you will easily find (do it!) that the answer is $t = \frac{v_0 - \sqrt{v_0^2 - 2gh}}{g}$, but let's see if we can do it from equation 9.2.9.

Let us take the origin to be the ground, and we shall measure distances y upward from the ground. The speed is then given by $v = +dy/dt$, so we use the positive sign in equation 9.2.9. The initial and final values of y are 0 and h respectively.

The total energy E is the initial kinetic energy $\frac{1}{2}mv_0^2$ (since the initial potential energy is zero), and the potential energy at height y is $V(y) = mgy$. Equation 9.2.9 therefore takes the form

$$t = +\sqrt{\frac{m}{2}}\int_0^h \frac{dy}{\sqrt{\frac{1}{2}mv_0^2 - mgy}}. \qquad 9.3.2$$

This yields the expected answer $t = \dfrac{v_0 - \sqrt{v_0^2 - 2gh}}{g}$. 9.3.3

iii. In this example, we shall have a stone falling from a height that is not negligible compared with the radius of the Earth, so that the acceleration is not constant. We shall suppose that we drop a stone from a point at a distance $r = b$ from the centre of the Earth, and we ask how long it will take to reach the surface of the Earth, radius a.

Let us take the origin to be the centre of the Earth, and we shall measure distances r radially outward from the centre. The speed is then given by $v = -dy/dt$, so we use the negative sign in equation 9.2.9. The initial and final values of r are b and a respectively.

The total energy E is the initial potential energy $-\dfrac{GMm}{b}$ (since the initial kinetic energy is zero), and the potential energy at distance r from the centre is $V(r) = -\dfrac{GMm}{r}$. Equation 9.2.9 therefore takes the form

$$t = -\sqrt{\frac{m}{2}}\int_b^a \frac{dr}{\sqrt{\frac{GMm}{r} - \frac{GMm}{b}}} \qquad 9.3.4$$

The gravity on the surface of Earth is $g_0 = \dfrac{GM}{a^2}$ so that equation 9.3.4 can be written

$$t = -\frac{1}{a}\sqrt{\frac{b}{2g_0}}\int_b^a \frac{dr}{\sqrt{\frac{b}{r} - 1}} \qquad 9.3.5$$

This can be integrated analytically to give

$$t = \frac{b}{a}\sqrt{\frac{b}{2g_0}}\left(\alpha + \tfrac{1}{2}\sin 2\alpha\right), \qquad 9.3.6$$

where $$\cos^2\alpha = a/b. \qquad 9.3.7$$

Here's a numerical example: How long would it take for a stone to fall to Earth from an initial height of 240,000 miles? In case the above method is too difficult, here's another way to do it - in your head in a few seconds!

240,000 miles is the radius of the Moon's orbit. The stone is falling in a highly elliptical orbit of major axis equal to the distance to the Moon - i.e. its semi major axis is half that of the Moon. Therefore by Kepler's third law, its period is equal to the Moon's period (which is 28 days) divided by $2\sqrt{2}$, which is 2.8. The orbital period of the stone is therefore 10 days. The time taken to drop to Earth is half of this, or five days. You might want to calculate it from equations 9.3.6 and 9.3.7 and see if you get the same answer!

(iv) What is the period of oscillation of a pendulum of length l swinging through an angle α?

The answer is that it is four times the time that it takes to rise from the vertical position to an angle α from the vertical. Thus we shall work out this time from equation 9.2.10 and multiply by four.

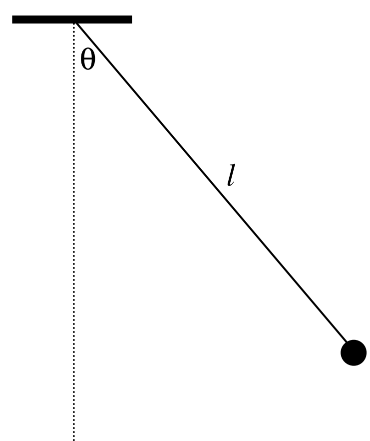


FIGURE IX.1

Let us adopt the upper end of the string as our level for zero potential energy. The potential energy $V(\theta)$ is then $-mgl\cos\theta$. The total energy E is equal to the potential energy when $\theta = \alpha$; that is to say, $E = -mgl\cos\alpha$. If we take the initial angle to be 0 and the final angle to be α, then the upward motion is such that $\omega = +d\theta/dt$, and we choose the positive sign for equation 9.2.10. The rotational inertia I is ml^2. Thus equation 9.2.10 gives

$$P = 4t = \sqrt{\frac{8l}{g}}\int_0^\alpha \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}}. \qquad 9.3.8$$

This can doubtless be expressed in terms of special functions that most of us are unfamiliar with, so you will probably opt to evaluate this numerically as a function of α. There is a small difficulty at the upper limit of the integration when the integrand becomes infinite. Indeed, in all cases in which the system is at rest at either the start or the finish, the denominator of equation 9.2.9 or 9.2.10 will necessarily be zero and hence the integrand will be infinite. In some cases, as in examples **(i)** to **(iii)**, the integral can be done analytically and there is no problem. If, however, as in the present example, the integration has to be done numerically, there is a potential problem and some ingenuity (often by making a change of variable) will have to be exercised.

Using a trigonometric identity, you can write equation 9.3.8 as

$$P = \sqrt{\frac{4l}{g}}\int_0^\alpha \frac{d\theta}{\sqrt{\sin^2\frac{1}{2}\alpha - \sin^2\frac{1}{2}\theta}}. \qquad 9.3.9$$

This doesn't get rid of the infinity, but now make a change of variable by letting

$$\sin\tfrac{1}{2}\theta = \sin\tfrac{1}{2}\alpha\sin\phi \qquad 9.3.10$$

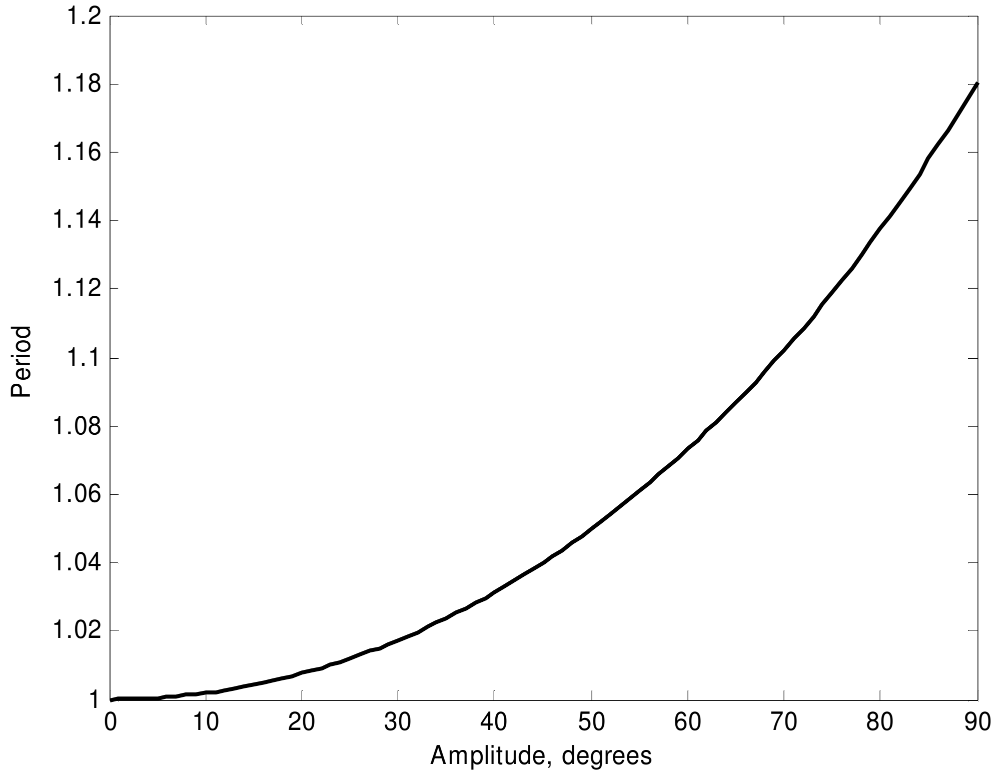
and the difficulty will disappear. In particular, the expression for the period becomes

$$P = 2\pi\sqrt{\frac{l}{g}} \times \frac{2}{\pi}\int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \sin^2\frac{1}{2}\alpha . \sin^2\phi}}. \qquad 9.3.11$$

Below is a graph of the period, in units of $2\pi\sqrt{\frac{l}{g}}$, versus α.

A further question might be asked. For example: What is the amplitude of the pendulum swing such that the period is 10 percent more than the small angle limit of $2\pi\sqrt{\frac{l}{g}}$? In other words, solve the equation $\frac{\sqrt{2}}{\pi}\int_0^\alpha \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}} = 1.1$ for α. This will be quite a challenge.

I make it $\alpha = 69^\circ.325146$



(v) A Semicircle, a Ring and a String

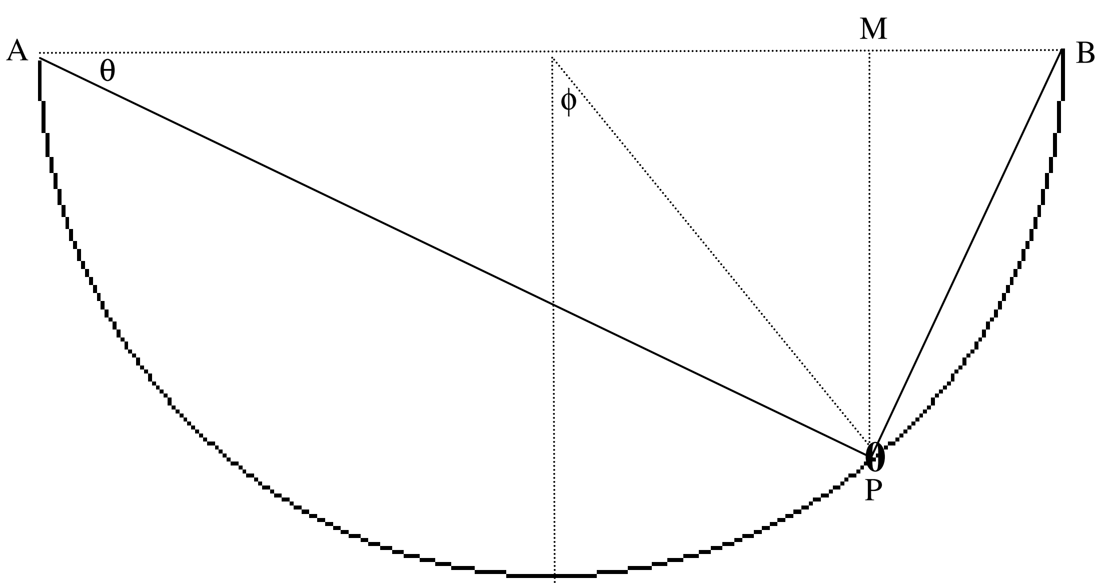


FIGURE IX.2

In this example, we have a smooth semicircular wire in a vertical plane. The radius of the semicircle is a. A ring of mass m at P can slide smoothly around the ring. An elastic string of natural length $2a$ is attached to the ends of the wire at A and B and is threaded through the ring. The force constant of the string is k. The ring is subjected to three conservative forces - gravity and the tensions in the two parts of the string. The ring is smooth, so there is no nonconservative friction.

By geometry the lengths of the following are

AP: $2a\cos\theta$

BP: $2a\sin\theta$

PM: $2a\sin\theta\cos\theta = a\sin 2\theta$

We are going to find the equilibrium position(s) of the ring, and see how long it takes to slide from one position on the semicircle to another.

Before doing any calculations, let's think about the physics qualitatively. Suppose that it is a very heavy ring and a weak string. In that case the ring will surely slide down to the bottom of the semicircle and stay there. On the other hand suppose that the ring is not very heavy but the string is quite strong. In that case we may well imagine that the ring may rest in stable equilibrium farther up the semicircle; indeed, if the string is very strong, the stable equilibrium position of the ring might be quite near the top. Of course, by the symmetry of the situation, there will always be an equilibrium position at the bottom of the semicircle, but, if the string is very strong, this position will be unstable, and, upon the slightest displacement, the ring will snap up to a higher position. Whether the position at the bottom of the semicircle will be stable or unstable depends on the relative strengths of the weight of the ring and the tension in the string. Let us then, in anticipation, refer to the ratio mg/ka by the symbol λ. In fact we shall find that if $\lambda > 0.586$, the position at the bottom of the semicircle will be stable, but if λ is less than this the positions of stable equilibrium will be higher up.

The extension of the string above its natural length is $2a(\cos\theta + \sin\theta - 1)$ and the depth of the ring below AB is $a\sin 2\theta$. Therefore, if we take AB to be the level for zero gravitational potential energy, the potential energy (elastic plus gravitational)) is

$$V(\theta) = \tfrac{1}{2}k\left[2a(\cos\theta + \sin\theta - 1)^2\right] - mga\sin 2\theta$$

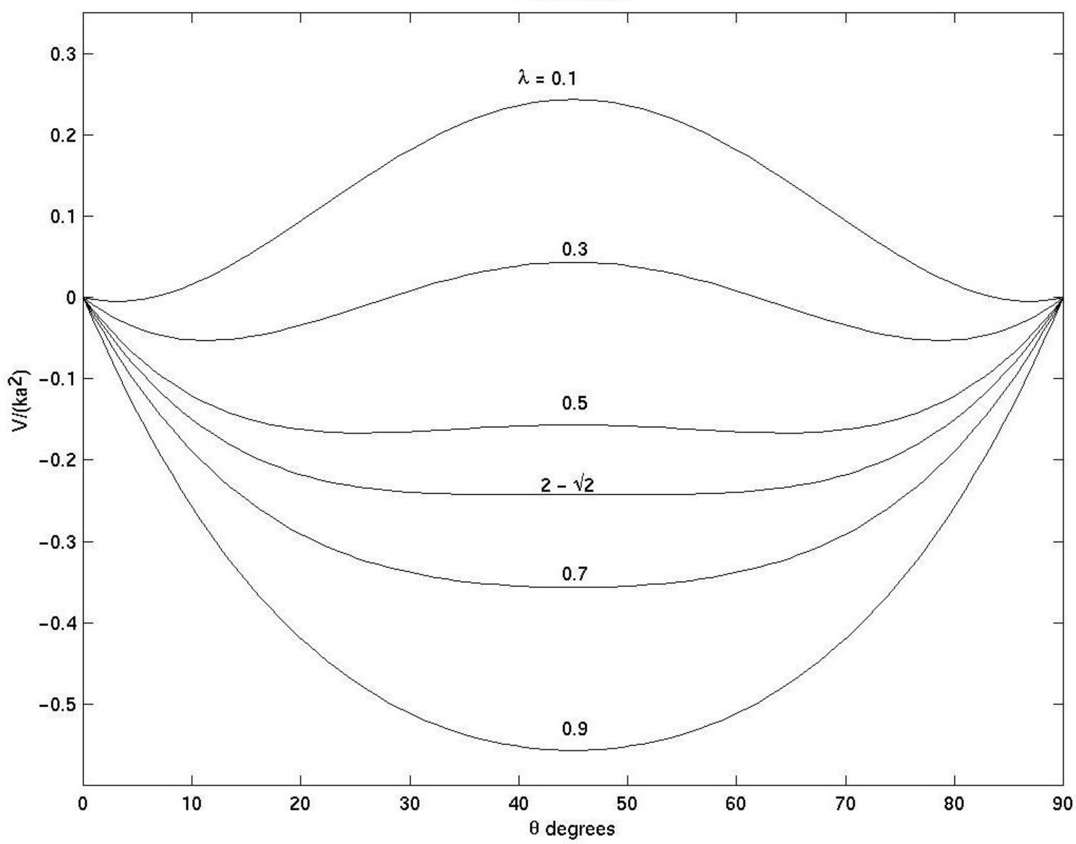
$$= ka^2\left[2\,(\cos\theta + \sin\theta - 1)^2 - \lambda\sin 2\theta\right] \qquad 9.3.11$$

Figure IX.3 shows this potential energy as a function of θ for several values of λ, including 0.586. We can see that, for λ greater than this, the position at the bottom of the ring ($\theta = 45^\circ$) is the only equilibrium position and it is stable. For small values of λ, the bottom, while an equilibrium position, is unstable, and there are two stable positions higher up. To calculate these equilibrium positions exactly, we need to determine where the derivative of V is zero, and to find whether these positions are stable or unstable, we need to examine the sign of the second derivative.

I leave it to the reader to work through the first derivative and to show that one condition for the derivative to be zero is for cos θ to equal sin θ; that is, $\theta = 45^0$, which corresponds to the bottom of the semicircle. Another condition for the first derivative to be zero is slightly more challenging to find, but you should find that the derivative is zero if

$$\sin\left(\theta + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2 - \lambda}. \qquad 9.3.12$$

FIGURE IX.3



This corresponds to a real value of θ only if $\lambda \le 2 - \sqrt{2} = 0.586$. The second derivatives are necessary to determine whether the equilibria are stable (V a minimum) or unstable (V a maximum) or a glance at figure IX.3 will be easier.

Now let's express the potential energy as a function U of the angle ϕ, so that $U(\phi) = V(\theta)$. From Figure IX.2, we see that $\phi = 90^{\circ} - 2\theta$. After some algebra and trigonometry, I find that the potential energy as a function of ϕ is given by

$$\frac{U(\phi)}{ka^2} = 4\cos^2\tfrac{1}{2}\phi - 4\sqrt{2}\cos\tfrac{1}{2}\phi - \lambda\cos\phi - \lambda - 4 + 4\sqrt{2}. \qquad 9.3.13$$

Now expand this carefully by Taylor's theorem as far as ϕ^2:

$$\frac{U(\phi)}{ka^2} = 2\left(\lambda + \sqrt{2} - 2\right)\phi^2. \qquad 9.3.14$$

What we have done is to approximate the potential energy function $U(\phi)$ by a parabola for small ϕ. Now a parabolic potential well is characteristic of simple harmonic motion. For example, in linear simple harmonic motion obeying the equation $m\ddot{x} = -kx$ the potential energy per unit mass is $\frac{1}{2}kx^2$ and the period is $2\pi\sqrt{\frac{m}{k}}\cdot$ In rotational simple harmonic motion obeying the equation $I\ddot{\theta} = -c\theta$ the potential energy per unit rotational inertia is $\frac{1}{2}c\theta^2$ and the period is $2\pi\sqrt{\frac{I}{c}}\cdot$ The rotational inertia here is just ma^2, so we find that the period of small oscillations is

$$P = \pi\sqrt{\frac{\lambda a}{\left(\lambda + \sqrt{2} - 2\right)g}}, \qquad 9.3.15$$

provided, of course, that $\lambda > 2 - \sqrt{2}$.

If you want to find how long the ring takes to slide from an initial position $\phi = \alpha$ to the bottom you can use equations 9.2.10 and 9.3.11, with $E = U(\alpha)$. You will find the usual difficulty that the integrand is zero at the start. I haven't actually tried the problem, because it looks slightly tedious, but I am fairly certain that it can be integrated analytically. If you do it and get an answer, make sure that, in the limit of small ϕ, you get the same as equation 9.3.15.

(vi) A rod of length $2l$ and mass m has one end freely pivoted on a horizontal floor. The rod is held at an initial angle of 45° to the vertical and then released. How long does it take for the rod to hit the floor? I'll leave you to work this one out. By the way, if I had started with the rod vertical, you would find that it takes an infinite time to fall - because the vertical position, although unstable, is an equilibrium position, and it would never get going unless given an infinitesimal displacement.

9.4 *Virtual Work*

We have seen that a mechanical system subject to conservative forces is in equilibrium when the derivatives of the potential energy with respect to the coordinates are zero. A method of solving such problems, therefore, is to write down an expression for the potential energy and put the derivatives equal to zero.

A very similar method is to use the principle of *virtual work*. In this method, we imagine that we act upon the system in such a manner as to increase one of the coordinates. We imagine, for example, what would happen if we were to stretch one of the springs, or to increase the angle between two jointed rods, or the angle that the ladder makes as it leans against the wall. We ask ourselves how much work we have to do on the system in order to increase this coordinate by a small amount. If the system starts from equilibrium, this work will be very small, and, in the limit of an infinitesimally small displacement, this "virtual work" will be zero. This method is very little different from setting the derivative of the potential energy to zero. I mention it here, however, because the concept might be useful in Chapter 13 in describing Hamilton's variational principle.

Let's start by doing a simple ladder problem by the method of virtual work. The usual uniform ladder of high school physics, of length $2l$ and weight mg, is leaning in limiting static equilibrium against the usual smooth vertical wall and the rough horizontal floor whose coefficient of limiting static friction is μ. What is the angle θ that the ladder makes with the vertical wall?

I have drawn the four forces on the ladder, namely: its weight mg; the normal reaction of the floor on the ladder, which must also be mg; the frictional force, which is μmg; and the normal (and only) reaction of the wall on the ladder, which must also be μmg.

There are several ways of doing this, which will be familiar to many readers. The only small reminder that I will give is to point out that, if you wish to combine the two forces at the foot of the ladder into a single force acting upwards and somewhat to the left, so that there are then just three forces acting on the ladder, the three forces must act through a single point, which will be above the middle of the ladder and to the right of the point of contact with the wall. But we are interested now in solving this problem by the principle of virtual work.

Before starting, I should warn that it is important in using the principle of virtual work to be meticulously careful about signs, and in that respect I remind readers that in the differential calculus the symbols δ and d in front of a scalar quantity x do not mean "a small change in" or "an infinitesimal change" in x. Such language is vague. The symbols stand for "a small *increase* in" and "an infinitesimal *increase* in".

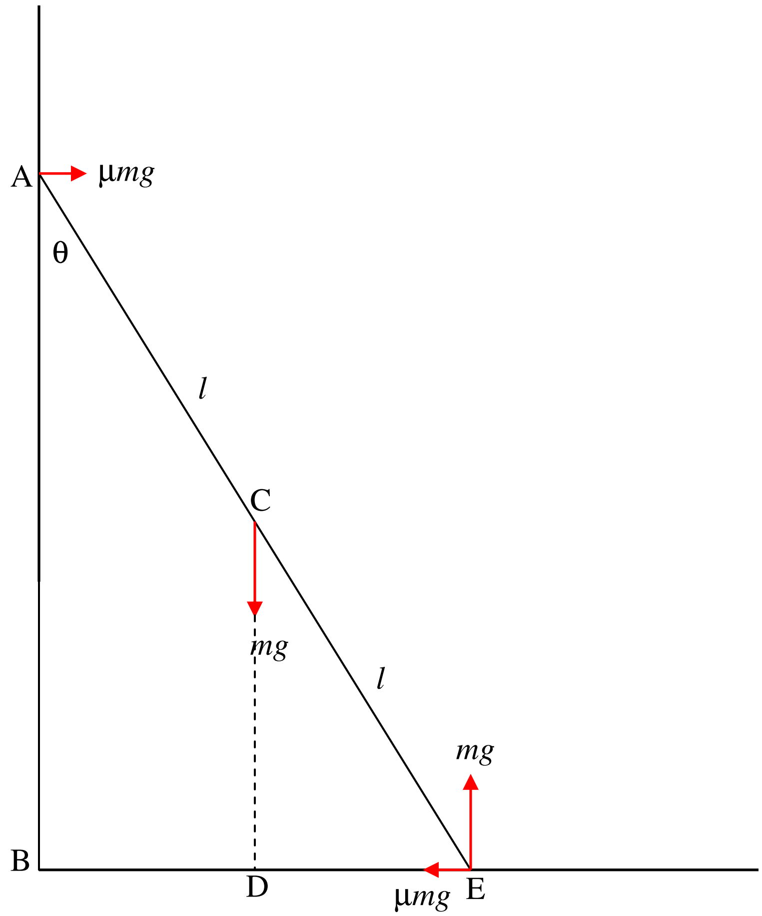


FIGURE IX.4

Let us take note of the following distances:

$$CD = l \cos \theta \qquad 9.4.1$$

and

$$BE = 2l \sin \theta . \qquad 9.4.2$$

If we were to **increase** θ by δθ, keeping the ladder in contact with wall and floor, the **increases** in these distances would be

$$\delta(CD) = -\, l \sin \theta \, \delta\theta \qquad 9.4.3$$

and

$$\delta(BE) = 2l \cos \theta \, \delta\theta . \qquad 9.4.4$$

Further, if were to **increase** θ by δθ, the work done **by** the force at C would be *mg* times the decrease of the distance CD, and the work done **by** the frictional force at E would be *minus* μ*mg* times the increase of the distance BE. The other two forces do no work. Thus the "virtual work" done **by** the external forces **on** the ladder is

$$mg.l\sin\theta\,\delta\theta \;-\; \mu mg.2l\cos\theta\,\delta\theta. \qquad 9.4.5$$

On putting the expression for the virtual work to zero, we obtain

$$\tan\theta \;=\; 2\mu\,. \qquad 9.4.6$$

You should verify that this is the same answer as you get from other methods – the easiest of which is probably to take moments about E.

There is something about virtual work which reminds me of thermodynamics. The first law of thermodynamics, for example is $\Delta U \;=\; \Delta q \;+\; \Delta w$, where ΔU is the **increase** of the internal energy of the system, Δq is the heat added **to** the system, and Δw is the work done **on** the system. Prepositions play an important part in thermodynamics. It is always mandatory to state clearly and without ambiguity whether work is done **by** the piston **on** the gas, or **by** the gas **on** the system; or whether heat is gained **by** the system or lost **from** it. Without these prepositions, all discussion is meaningless. Likewise in solving a problem by the principle of virtual work, it is always essential to say whether you are describing the work done **by** a force **on** what part of the system (on the ladder or on the floor?) and whether you are describing an increase or a decrease of some length or angle.

Let us move now to a slightly more difficult problem, which we'll try by three different methods – including that of virtual work.

In figure IX.5, a uniform rod AB of weight Mg and length $2a$ is freely hinged at A. The end B carries a smooth ring of negligible mass. A light inextensible string of length l has one end attached to a fixed point C at the same level as A and distant $2a$ from it. It passes through the ring and carries at its other end a weight $\frac{1}{10}Mg$ hanging freely. (The "smooth" ring means that the tension in the string is the same on both sides of the ring.) Find the angle CAB when the system is in equilibrium.

I have marked in various angles and lengths, which can easily be determined from the geometry of the system, and I have also marked the four forces on the rod.

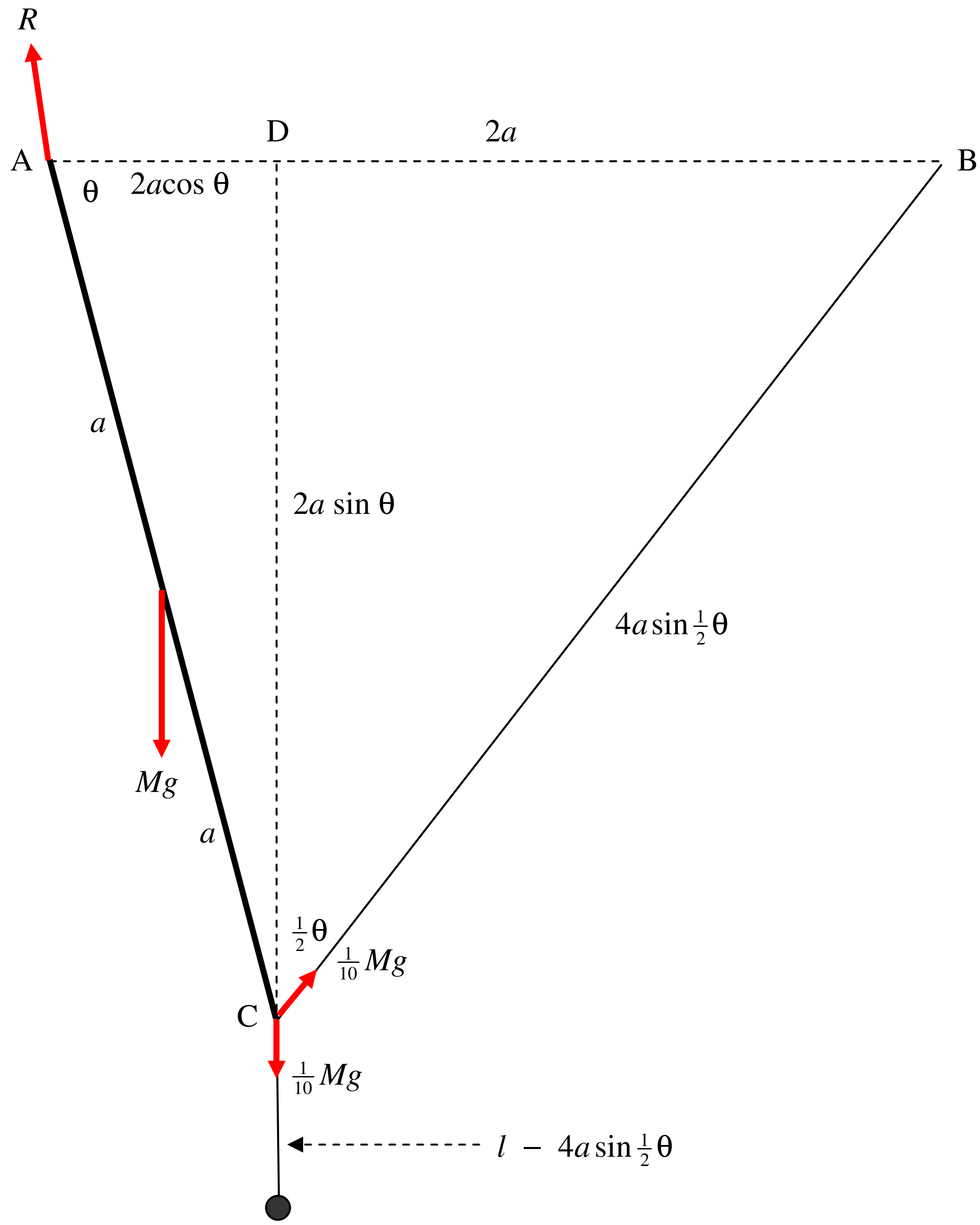


FIGURE IX.5

Let us first try a very conventional method. We know rather little about the force **R** of the hinge on the rod (though see below), and therefore this is a good reason for taking moments about the point A. We immediately obtain

$$Mga\cos\theta + \tfrac{1}{10}Mg.2a\cos\theta = \tfrac{1}{10}Mg.2a\cos\tfrac{1}{2}\theta. \qquad 9.4.7$$

Divide by Mga and set $\cos\theta = 2c^2 - 1$, where $c = \cos\frac{1}{2}\theta$. After a little algebra, we obtain $12c^2 - c - 6 = 0$ and hence we find for the equilibrium condition that $\theta = 82^{\circ}\,49'$ or $263^{\circ}\,37'$.

The latter, by the way, is a physically valid solution – you might want to sketch it.

I pointed out that the fact that we knew little about the reaction **R** at the hinge was a good reason for taking moments about A – although in fact I have drawn **R** in about the right direction. If you replace the two forces at the ring with a single force, this single force will bisect the angle between the two portions of the string. There are then three forces on the rod, and they must be concurrent at a point. That's how I knew the direction of **R**. However, now that we know some angles, it would be a simple matter, if need be, to find the horizontal and vertical components of **R**.

Now let's try the same problem using energy conditions. We'll take the zero of potential energy when the rod is horizontal – at which time the small mass is at a distance l below the level AC.

When the angle CAB = θ, the distance of the centre of mass of the rod below AC is $a \sin\theta$ and the distance of the small mass below AC is $l - 4a\sin\frac{1}{2}\theta + 2a\sin\theta$, so that the potential energy is

$$V = -Mga\sin\theta + \tfrac{1}{10}Mg[l - (l - 4a\sin\tfrac{1}{2}\theta + 2a\sin\theta)] = -\tfrac{2}{5}Mga(3\sin\theta - \sin\tfrac{1}{2}\theta). \qquad 9.4.8$$

The derivative is $$\frac{dV}{d\theta} = -\tfrac{2}{3}Mga(3\cos\theta - \tfrac{1}{2}\cos\tfrac{1}{2}\theta), \qquad 9.4.9$$

and setting this to zero will produce the same results as before. Further differentiation (do it), or a graph of $V : \theta$ (do it), will show that the $82^{\circ}\ 49'$ solution is stable and the $263^{\circ}\ 37'$ solution is unstable.

Now let's try it by virtual work. We are going to increase θ by δθ and see how much work is done.

The distance of the centre of mass of the rod below AC is $a \sin\theta$, and if θ increases by δθ, this will increase by $a \cos\theta\,\delta\theta$, and the work done by Mg will be $Mga \cos\theta\,\delta\theta$.

The distance of the ring below AC is $2a \sin\theta$, and if θ increases by δθ, this will increase by $2a \cos\theta\,\delta\theta$, and the work done by the downward force will be $\frac{1}{10}Mg.2a\cos\theta\,\delta\theta$.

The distance BC is $4a\sin\frac{1}{2}\theta$, and if θ increases by δθ, this will increase by $2a\cos\frac{1}{2}\theta\,\delta\theta$, and the work done by the sloping force will be ***MINUS*** $\frac{1}{10}Mg.2a\cos\frac{1}{2}\theta\,\delta\theta$.

Thus the virtual work is

$$Mg.a\cos\theta\,\delta\theta + \tfrac{1}{10}Mg.2a\cos\theta\,\delta\theta - \tfrac{1}{10}Mg.2a\cos\tfrac{1}{2}\theta\,\delta\theta. \qquad 9.4.10$$

If we put this equal to zero, we obtain the same result as before.