CHAPTER 15
SPECIAL RELATIVITY

15.1. *Introduction*

Why a chapter on relativity in a book on "classical mechanics"? A first excuse might be that the phrase "classical mechanics" is used by different authors to mean different things. To some, it means "pre-relativity"; to others it means "pre-quantum mechanics". For the purposes of this chapter, then, I mean the latter, so that special relativity may fairly be included in "classical" mechanics. A second excuse is that, apart from one brief foray into an electromagnetic problem, this chapter deals only with mechanical, kinematic and dynamical problems, and therefore deals with only a rather restricted part of relativity that can be dealt with conveniently in a single chapter of classical mechanics rather than in a separate book. This is in fact a quite substantial restriction, because electromagnetic theory plays a major role in special relativity. It was in fact difficulties with electromagnetic theory that led Einstein to the special theory of relativity. Indeed, Einstein's theory of relativity was introduced to the world in a paper with the title *Zur Elektrodynamik bewegter Körper* (*On the Electrodynamics of Moving Bodies*), *Annalen der Physik*, **17**, 891 (1905).

The phrase "special" relativity deals with the transformations between reference frames that are moving with respect to each other at constant relative velocities. Reference frames that are accelerating or rotating or moving in any manner other than at constant speed in a straight line are included as part of general relativity and are not considered in this chapter.

15.2. *The Speed of Light*

The speed of light is, by definition, exactly $2.997\ 924\ 58 \times 10^8$ m s^{-1}, and is the same relative to all observers.

This seemingly simple sentence invites several comments.

First: Note that I have used the word "speed". Some writers use the word "velocity" as if it were merely a more impressive and scientific-sounding synonym for "speed". I trust that all readers of these notes know the difference and will use the word "speed" when they mean "speed", and the word "velocity" when they mean "velocity – surely not an unreasonable demand. To say that the "velocity" of light is the same for all observers means that the direction of travel of light is the same relative to all observers. This is doubtless not at all what a writer who uses the word "velocity" intends to convey – but it is the literal (and of course quite erroneous) meaning of the assertion.

Second: How can we possibly *define* the speed of light to have a certain *exact* value? Surely the speed of light is what we find it to be, and we are not free to *define* its value. But in fact we *are* allowed to do this, and the explanation, briefly, is as follows.

Over the course of history, the *metre* has been defined in several different ways. At one time it was a specified fraction of the circumference of Earth. Later, it was the distance between two scratches on a bar of platinum-iridium alloy held in Paris. Later still it was a specified number of wavelengths of a particular line in the spectrum of mercury, or cadmium, or argon or krypton. In our present state of technology it is far easier to measure and reproduce precise standards of *frequency* than it is to measure and reproduce standards of *length*. Because of that, the current SI (Système International) unit of time is the SI second, which is based on the frequency of a particular transition in the spectrum of caesium, and from there, the metre is *defined* as the distance travelled by light *in vacuo* in a defined fraction of an SI second, the speed of light being assigned the exact value quoted above.

Detailed discussion of the exact definitions of the units of time, distance and speed is part of the subject of *metrology*. That is an important and interesting subject, but it is only marginally relevant to the topic of relativity, and consequently, having quoted the exact value of the speed of light, we leave further discussion of metrology here.

Third: How can the speed of light be the same relative to *all observers*? This assertion is absolutely central to the theory of special relativity, and it may be regarded as its fundamental and most important principle. We shall discuss it further in the remainder of the chapter.

15.3. *Preparation*

The ratio of the speed υ of a body (or a particle, or a reference frame) to the speed of light is often given the symbol β:

$$\beta = \upsilon / c. \qquad 15.3.1$$

For reasons that will become apparent (I hope!) later, the range of β is usually restricted to between 0 and 1. In our study of special relativity, we shall find that we have to make frequent use of a number of functions of β. The most common of these are

$$\gamma = (1-\beta^2)^{-1/2}, \qquad 15.3.2$$

$$k = \sqrt{(1+\beta)/(1-\beta)}\ , \qquad 15.3.3$$

$$z = k - 1\ , \qquad 15.3.4$$

$$K = \gamma - 1\ , \qquad 15.3.5$$

$$\phi = \tfrac{1}{2}\ln[(1+\beta)/(1-\beta)] = \tanh^{-1}\beta = \ln k\ . \qquad 15.3.6$$

$$\theta = \cos^{-1}\gamma = \sin^{-1}(i\beta\gamma). \qquad 15.3.7$$

In figures XV.1-3 I draw γ, k and ϕ as functions of β. The functions γ and k go from 1 to ∞ as β goes from 0 to 1; z, K and ϕ go from 0 to ∞. The function θ is imaginary.

Many – one might even say most – problems in special relativity (including examination and homework questions!) amount, when stripped of their verbiage, to the following:

"Given one of the quantities β, γ, k, z, K, ϕ, θ, calculate one of the others." Thus I would suggest that, even before you have any idea what these quantities mean, you might write a program for your computer (or programmable calculator) such that, when you enter any one of the real quantities, the computer will instantly return all seven of them. This will save you, on future occasions, from having to remember the exact formulas or having to bother with tedious arithmetic, so that you can concentrate your mind on understanding the relativity.

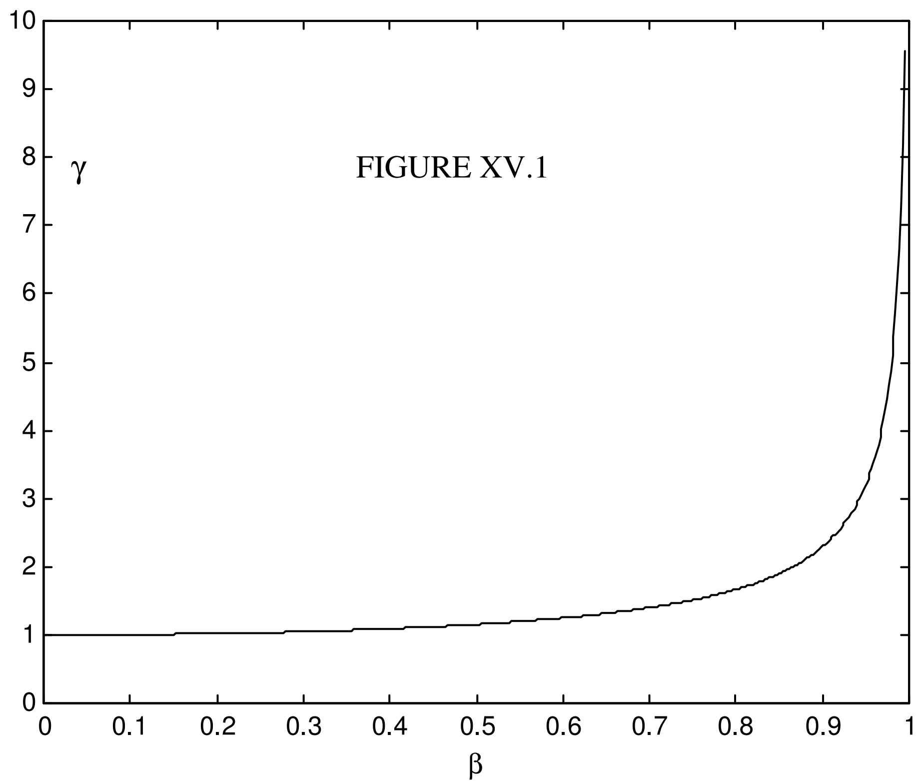


FIGURE XV.1

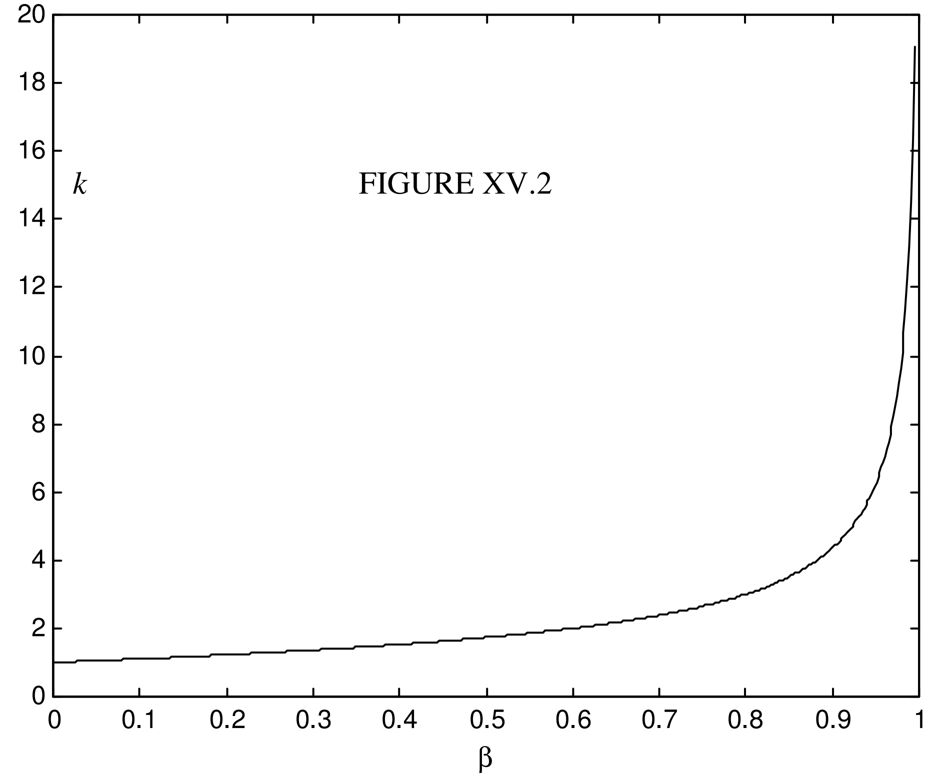


FIGURE XV.2

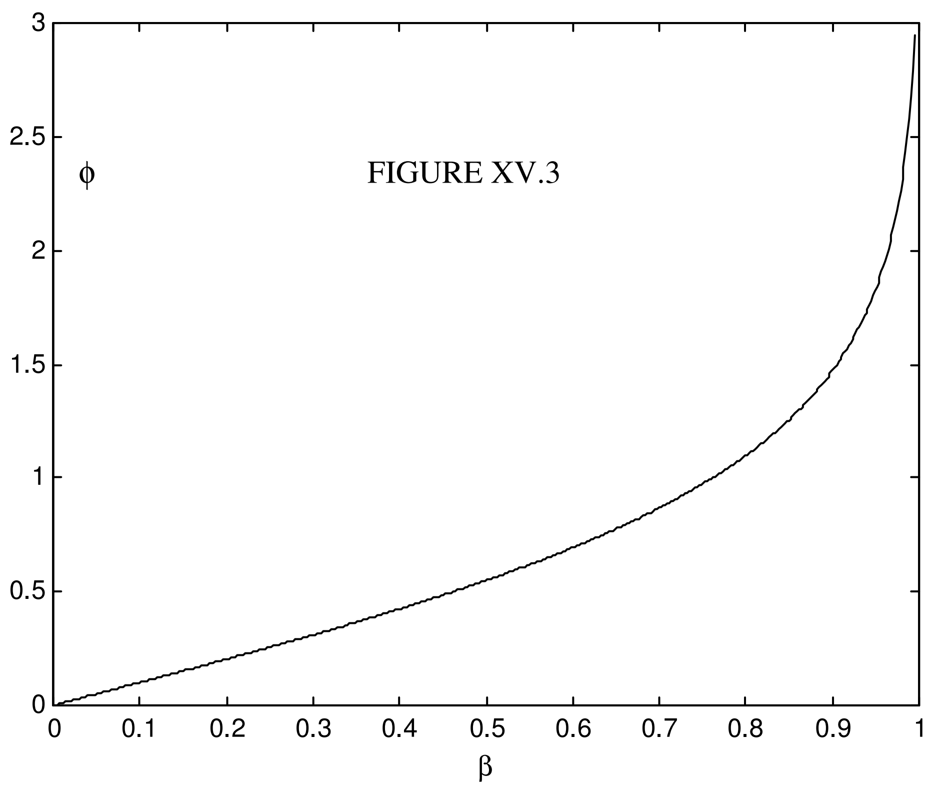


FIGURE XV.3

Just for future reference, I tabulate here the relations between these various quantities. This has involved some algebra and typesetting; I don't think there are any mistakes, but I hope some reader might check through them all carefully and will let me know (jtatum at uvic dot ca) if he or she finds any.

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} = \frac{k^2 - 1}{k^2 + 1} = \frac{z(z+2)}{(z+1)^2 + 1} = \frac{\sqrt{K(K+2)}}{K+1} = \tanh\phi \text{ or } \frac{e^{2\phi} - 1}{e^{2\phi} + 1} = -i\tan\theta$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{k^2 + 1}{2k} = \frac{(z+1)^2 + 1}{2(z+1)} = K + 1 = \cosh\phi \ \textit{or}\ \tfrac{1}{2}(e^{\phi} + e^{-\phi}) = \cos\theta$$

$$k = \sqrt{\frac{1+\beta}{1-\beta}} = \gamma + \sqrt{\gamma^2 - 1} = z + 1 = K + 1 + \sqrt{K(K+2)} = e^{\phi} = e^{-i\theta}$$

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 = \gamma - 1 + \sqrt{\gamma^2 - 1} = k - 1 = K + \sqrt{K(K+2)} = e^{\phi} - 1 = e^{-i\theta} - 1$$

$$K = \frac{1}{\sqrt{1-\beta^2}} - 1 = \gamma - 1 = \frac{(k-1)^2}{2k} = \frac{z^2}{2(z+1)} = \frac{(e^{\phi} - 1)^2}{2e^{\phi}} = \cos\theta - 1$$

$$\phi = \tanh^{-1}\beta \text{ or } \tfrac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right) = \cosh^{-1}\gamma \text{ or } \ln\left(\gamma + \sqrt{\gamma^2 - 1)}\right) = \ln k = \ln(z+1) = \ln\left(K + 1 + \sqrt{K(K+2)}\right) = -i\theta$$

$$\theta = \frac{i}{2}\ln\left(\frac{1+\beta}{1-\beta}\right) = i\ln\left(\gamma + \sqrt{\gamma^2 - 1}\right) = i\ln k = i\ln\left(z+1\right) = i\ln\left[K + 1 + \sqrt{K(K+2)}\right] = i\phi$$

15.4. *Speed is Relative. The Fundamental Postulate of Special Relativity.*

You are sitting in a railway carriage (or a railroad car, if you prefer the term). The windows and curtains are closed and you cannot see outside. You are asked to measure the constant speed of the carriage along its tracks. You try a number of experiments. You measure the period of a simple pendulum. You slide a puck and roll a ball down an inclined plane. You throw a ball vertically up in the air and catch it as it comes down. You throw it up at an angle and you watch it describe a graceful parabola. You cause billiard balls to collide on the billiards table thoughtfully provided in your carriage. You experiment with a torsion pendulum. You stand a pencil on its end and you watch it as it falls to a horizontal position.

All your careful work is to no avail. None of them tells you what speed you are moving at, or even if you are moving at all. After exhausting all mechanical experiments you can think of, you are led to the conclusion:

It is impossible to determine the speed of motion of a uniformly-moving reference frame by means of any mechanical experiment performed within that frame.

Frustrated, you open a curtain on one side of the carriage. You look out and you see that there is another train on the line next to you. It appears to be moving backwards. Or are you moving forwards? Or are you both moving in the same direction but at different speeds? You still can't tell.

You move to the other side of the carriage and open the curtain there. This time you see the station platform, and the station platform is moving backwards. Or are you moving forwards? (Those of you who have not done much travel by train may not appreciate just how very strong the impression can be that the platform is moving.) What does it mean, anyway, to say that it is you that is moving rather than the platform?

The following story is not true, but it ought to be. (It is an "apocryphal" story.) Einstein was travelling by train across Canada. Halfway across the Prairies he leant across and tapped on the knee of his fellow passenger and asked: "Excuse me, mein Herr, bitte, but does Regina stop at this train?"

You are about to conclude that it is not possible by any means, whether by experiment or by observation, to determine the speed of your reference frame, or even whether it is moving or stationary.

But not so hasty! I am about to invent a speedometer, which I intend to patent and to use to make myself rich. I am going to use my invention to measure the speed of our train – without even looking out of the window!

We shall set up two long parallel glass rods in the middle of the corridor, parallel to the railway lines and to the velocity of the train. We shall suspend the rods horizontally, side by side from a common support, and we shall rub each of them with a silken handkerchief, so that each of them bears an electrostatic charge of λ C m^{-1}. They will repel each other with an electrostatic force per unit length of

$$F_{\rm e} = \frac{\lambda^2}{4\pi\epsilon_0 r} \quad \text{N m}^{-1}, \qquad 15.4.1$$

where r is their distance apart, and consequently they will hang out of the vertical – see figure XV.4.

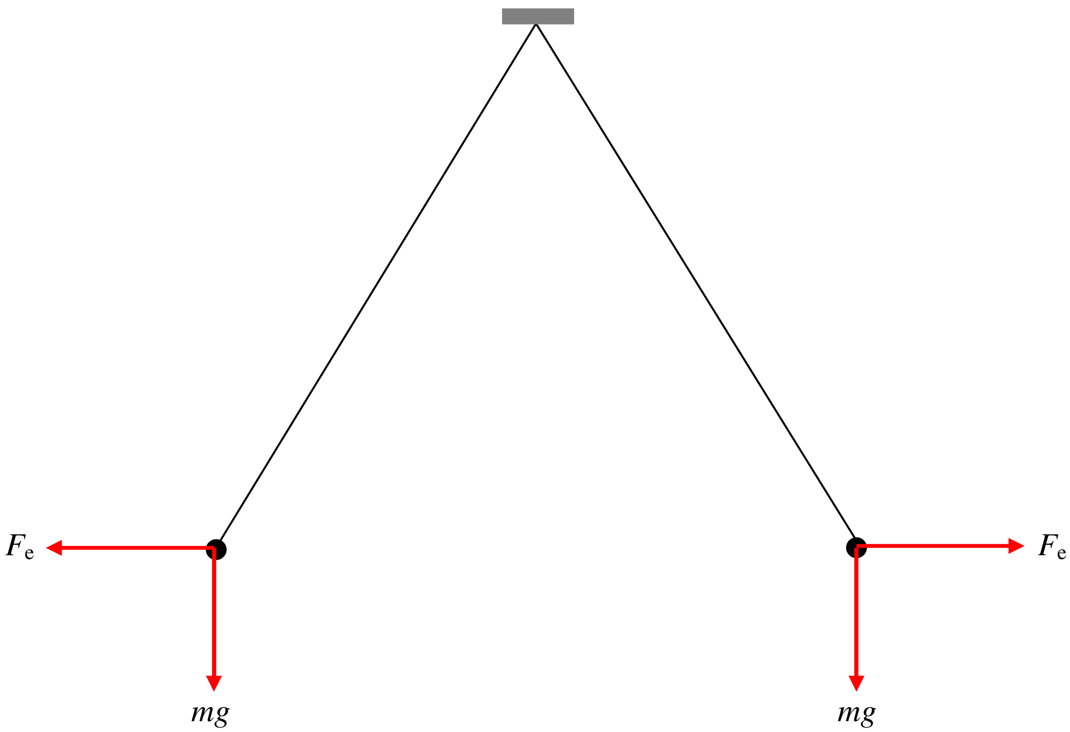


FIGURE XV.4

Now see what happens when the train moves forward at speed υ. Each rod, bearing a charge λ per unit length, is now moving forward at speed υ, and therefore each rod constitutes an electric current $\lambda\upsilon$ A. Therefore, by Ampère's law, in addition to the Coulomb repulsion, they will experience a magnetic attraction per unit length equal to

$$F_{\rm m} = \frac{\mu_0 \lambda^2 \upsilon^2}{4\pi r} \text{ N m}^{-1} . \qquad 15.4.2$$

The net repulsive force per unit length is now

$$\frac{\lambda^2}{4\pi\epsilon_0 r^2}\left(1 - \mu_0 \epsilon_0 \upsilon^2\right). \qquad 15.4.3$$

This is a little less than it was when the train was stationary, so the angle between the suspending strings is a little less, as shown in figure XV.5. It might be noted that the force between the strings is reduced to zero (and the angle also becomes zero) when the train is travelling at a speed $1/\sqrt{\mu_0 \epsilon_0}$. We remember from electromagnetic theory that the permeability of free space is $\mu_0 = 4\pi \times 10^{-7}$ H m^{-1} and that the permittivity ϵ_0 is

8.8542×10^{-12} F m^{-1}; consequently the force and the angle drop to zero and the strings hang vertically, when the train is moving at a speed of 2.998×10^{8} m s^{-1}.

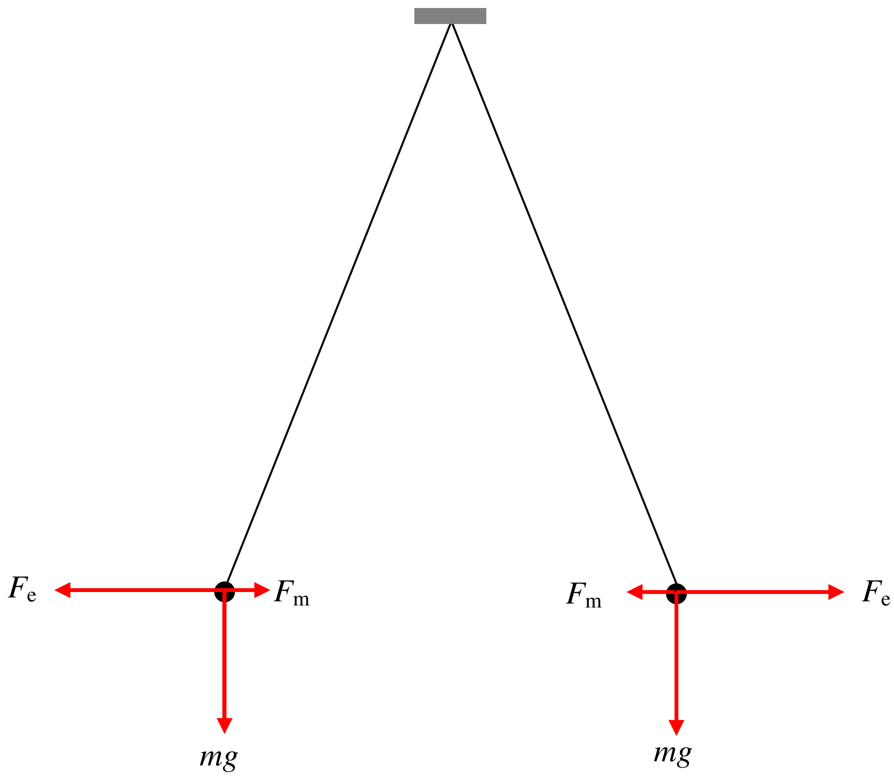


FIGURE XV.5

To complete my invention, I am now going to attach a protractor to the instrument, but instead of marking the protractor in degrees, I am going to calibrate it in miles per hour, and my speedometer is now ready for use (figure XV.6).

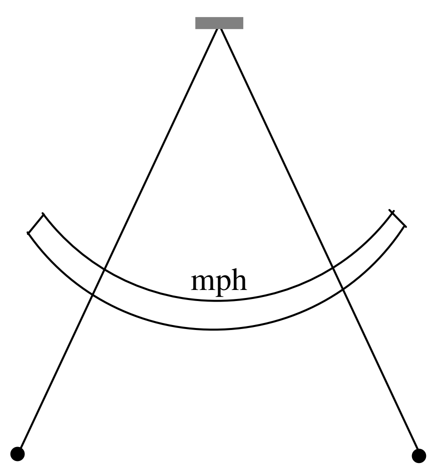


FIGURE XV.6

You now have a choice. Either:

i. You can choose to believe that the speedometer will work and you can accompany me to the patent office to see if they will grant a patent for this invention, which will measure the speed of a train without reference to any external reference frame. If you choose to believe this, there is no need for you to read the remainder of the chapter on special relativity.

Or:

ii. You can say that it defies common sense to believe that it is possible to determine whether a given reference frame is moving or stationary, let alone to determine its speed. Common sense dictates that

It is impossible to determine the speed of motion of a uniformly-moving reference frame by any means whatever, whether by a mechanical or electrical or indeed any experiment performed entirely or partially within that frame, or even by reference to another frame.

Your common sense, then, leads you – as it should – to the fundamental principle of special relativity. Whereas some people protest that relativity "defies common sense", in fact relativity *is* common sense, and its predictions (such as your prediction that my speedometer will not work) are exactly what common sense would lead you to expect.

15.5. *The Lorentz Transformations*

For the remainder of this chapter I am taking, as a fundamental postulate, that

It is impossible to determine the speed of motion of a uniformly-moving reference frame by any means whatever, whether by a mechanical or electrical or indeed any experiment performed entirely or partially within that frame, or even by reference to another frame

and consequently I am choosing to believe that my speedometer will not work. If it is impossible by any electrical experiment to determine our speed, we must assume that all the electromagnetic equations that we know, not just the ones that we have quoted, but indeed Maxwell's equations, which embrace all electromagnetic phenomena, are the same in all uniformly-moving reference frames.

One of the many predictions of Maxwell's equations is that electromagnetic radiation (which includes light) travels at a speed

$$c = 1/\sqrt{\mu_0 \varepsilon_0}\,. \qquad 15.5.1$$

Presumably neither the permeability nor the permittivity of space changes merely because we believe that we are travelling through space – indeed it would defy common sense to suppose that they would. Consequently, our acceptance of the fundamental principle of special relativity is equivalent to accepting as a fundamental postulate that the speed of light *in vacuo* is the same for all observers in uniform relative motion. We shall take anything other than this to be an outrage against common sense – though acceptance of the principle will require a careful examination of our ideas concerning the relations between time and space.

Let us imagine two reference frames, Σ and Σ'. Σ' is moving to the right (positive x-direction) at speed υ relative to Σ. (For brevity, I shall from time to time refer to Σ as the “stationary” frame, in the hope that this liberty will not lead to misunderstanding.) At time $t = t' = 0$ the two frames coincide, and at that instant someone strikes a match at the common origin of the two frames. At a later time, which I shall call t if referred to the frame Σ, and t' if referred to Σ', the light from the match forms a spherical wavefront travelling radially outward at speed c from the origin O of Σ, and the equation to this wavefront, when referred to the frame Σ, is

$$x^2 + y^2 + z^2 - c^2t^2 = 0. \qquad 15.5.2$$

Referred to Σ', it also travels outward at speed c from the origin O' of Σ', and the equation to this wavefront, when referred to the frame Σ', is

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = 0. \qquad 15.5.3$$

Most readers will accept, I think, that $y = y'$ and $z = z'$. Some formal algebra may be needed for a rigorous proof, but that would distract from our main purpose of finding a transformation between the primed and unprimed coordinates such that

$$x'^2 - c^2t'^2 = x^2 - c^2t^2. \qquad 15.5.4$$

It is easy to show that the “Galilean” transformation $x' = x - ct,\ t' = t$ does not satisfy this equality, so we shall have to try harder.

Let us seek linear transformations of the form

$$x' = Ax + Bt, \qquad 15.5.5$$

$$t' = Cx + Dt, \qquad 15.5.6$$

which satisfy equation 15.5.4.

We have
$$\frac{x'}{t'} = \frac{Ax + Bt}{Cx + Dt}, \qquad 15.5.7$$

and, by inversion,
$$\frac{x}{t} = \frac{Dx' - Bt'}{-Cx' + At'}. \qquad 15.5.8$$

Consider the motion of O' relative to Σ and to Σ'. We have $x/t = \upsilon$ and $x' = 0$.

$$\therefore \qquad \upsilon = -B/A. \qquad 15.5.9$$

Consider the motion of O relative to Σ' and to Σ. We have $x'/t' = -\upsilon$ and $x = 0$.

$$\therefore \qquad -\upsilon = B/D. \qquad 15.5.10$$

From these we find that $D = A$ and $B = -A\upsilon$, so we arrive at

$$x' = A(x - \upsilon t) \qquad 15.5.11$$

and
$$t' = Cx + At. \qquad 15.5.12$$

On substitution of equations 15.5.11 and 15.5.12 into equation 15.5.4, we obtain

$$A^2(x - \upsilon t)^2 - c^2(Cx + At)^2 = x^2 - c^2t^2. \qquad 15.5.13$$

Equate powers of t^2 to obtain

$$A = \frac{1}{\sqrt{1 - \upsilon^2/c^2}} = \gamma. \qquad 15.5.14$$

Equate powers of xt to obtain

$$C = -\frac{\upsilon\gamma}{c^2}. \qquad 15.5.15$$

Equating powers of x^2 produces no new information.

We have now determined A, B, C and D, and we can substitute them into equations 15.5.5 and 15.5.6, and hence we arrive at

$$x' = \gamma(x - \upsilon t) \qquad 15.5.16$$

and $$t' = \gamma(t - vx/c^2) \ . \qquad 15.5.17$$

These, together with $y = y'$ and $z = z'$, constitute the *Lorentz transformations*, which, by suitable choice of axes, guarantee the invariance of the speed of light in all reference frames moving at constant velocities relative to one another.

To express x and t in terms of x' and t', you may, if you are good at algebra, solve equations 15.5.16 and 15.5.17 simultaneously for x' and t', or, if instead, you have good physical insight, you will merely reverse the sign of v and interchange the primed and unprimed quantities. Either way, you should obtain

$$x = \gamma(x' + vt') \qquad 15.5.18$$

and $$t = \gamma(t' + vx'/c^2) \ . \qquad 15.5.19$$

15.6. *But This Defies Common Sense*

At this stage one may hear the protest: "But this defies common sense!". One may hear it again as we encounter several predictions of the invariance of the speed of light and of the Lorentz transformations. But, if you have read this far, it is too late to make such protest. You have already, at the end of Section 15.4, made your choice, and you then decided that it defies common sense to suppose that one can somehow determine the speed of a reference frame by some experiment or observation. You rejected that notion, and it was the *application* of common sense, not its abandonment, that led us into the Lorentz transformations and the invariance of the speed of light.

There may be other occasions when we are tempted to protest "But this defies common sense!", and it is therefore always salutary to recall this. For example, we shall later learn that if a train is moving at speed V relative to the station platform, and a passenger is walking towards the front of the train at a speed v relative to the train, then, relative to the platform, he is moving at a speed just a little bit less that $V + v$. When we protest, we are often presented with an "explanation" along the following lines:

In every day life, trains do not move at speeds comparable to the speed of light, nor do walking passengers. Therefore, we do not notice that the combined speed is a little bit less than $V + v$. After all, if $V = 60$ mph and $v = 4$ mph, the combined speed is $0.999\,999\,999\,999\,999\,5 \times 64$ mph. The formula $V + v$ is just an *approximation*, we are told, and we have the erroneous impression that the combined speed is exactly $V + v$ only because we are accustomed, in daily life, to experiencing speeds that are small compared with the speed of light.

This explanation somehow does not seem to be satisfactory – and nor should it, for it is *not* a correct explanation. It seems to be an explanation invented for the benefit of the nonscientific layman – but nothing is ever made easy to understand by giving an incorrect explanation under the pretence of "simplifying" something. It is *not* correct merely to say that the Galilean transformations are just an "approximation" to the "real" transformations.

The problem is that it is exceedingly difficult – perhaps impossible – to describe exactly what is meant by "distance" and "time interval". It is almost as difficult as describing colours to a blind person, or even describing your sensation of the colour red to another seeing person. We have no guarantee that every person's perception of colour is the same. The best that can be done to describe what we mean by distance and time interval is to *define* how distances and times *transform* between reference frames. The Lorentz transformations, which we have adopted in order to make it meaningless to discuss the absolute velocity of a reference frame, amount to a useful *working definition* of the meanings of space and time. Once we have adopted this definition, "common sense" no longer comes into the matter. There is no longer a *mystery* which our minds cannot quite grasp; from this point on it merely becomes a matter of algebra as to how a measurement of length or of time interval, or of speed, or of mass, as appropriately defined, transforms when referred to one reference or to another. There is no impossible feat of imagination to be done.

15.7 *The Lorentz Transformation as a Rotation*

The Lorentz transformation can be written

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \qquad 15.7.1$$

where $x_1 = x$, $x_2 = y$, $x_3 = z$ and $x_4 = -ict$, and similarly for primed quantities. Please don't just take my word for this; multiply the matrices, and verify that this equation does indeed represent the Lorentz transformation. You could, if you wish, also write this for short:

$$\mathbf{x'} = \boldsymbol{\lambda}\mathbf{x} . \qquad 15.7.2$$

Another way of writing the Lorentz transformation is

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_0' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_0 \end{pmatrix}, \qquad 15.7.3$$

where $x_1 = x$, $x_2 = y$, $x_3 = z$ and $x_0 = ct$, and similarly for primed quantities.

Some people prefer one version; others prefer the other. In any case, a set of four quantities that transforms like this is called a 4-*vector*. Those who dislike version 15.7.1 dislike it because of the introduction of imaginary quantities. Those who like version 15.7.1 point out that the expression $[(\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2 + (\Delta x_4)^2]^{1/2}$ (the "interval" between two events) is invariant in four-space – that is, it has the same value in all uniformly-moving reference frames, just as the distance between two points in three-space, $[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2]^{1/2}$, is independent of the position or orientation of any reference frame. In version 15.7.3, the invariant interval is $[(\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2 - (\Delta x_0)^2]^{1/2}$. Those who prefer version 15.7.1 dislike the minus sign in the expression for the interval. Those who prefer version 15.7.3 dislike the imaginary quantities of version 15.7.1.

For the time being, I am going to omit *y* and z, so that I can concentrate my attention on the relations between *x* and *t*. Thus I am going to write 15.7.1 as

$$\begin{pmatrix} x_1' \\ x_4' \end{pmatrix} = \begin{pmatrix} x' \\ ict' \end{pmatrix} = \begin{pmatrix} \gamma & i\beta\gamma \\ -i\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x \\ ict \end{pmatrix} \qquad 15.7.4$$

and equation15.7.3 as

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix} \qquad 15.7.5$$

Readers may notice how closely equation 15.7.4 resembles the equation for the transformation of coordinates between two reference frames that are inclined to each other at an angle. (See Celestial Mechanics Section 3.6.) Indeed, if we let $\cos\theta = \gamma$ and $\sin\theta = i\beta\gamma$, equation 15.7.4 becomes

$$\begin{pmatrix} x' \\ ict' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ ict \end{pmatrix} \qquad 15.7.6$$

The matrices in equations 15.7.1, 15.7.4 and 15.7.6 are orthogonal matrices and they satisfy each of the criteria for orthogonality described, for example, in Celestial Mechanics Section 3.7. We can obtain the converse relations (i.e. we can express *x* and *t* in terms of *x*' and *t*') by interchanging the primed and unprimed quantities and either reversing the sign of β or of θ or by interchanging the rows and columns of the matrix.

There is a difficulty in making the analogy between the Lorentz transformation as expressed by equation 15.7.4 and rotation of axes as expressed by equation 15.7.6 in that, since $\gamma > 1$, θ is an imaginary angle. (At this point you may want to reach for your ancient, brittle, yellowed notes on complex numbers and hyperbolic functions.) Thus $\theta = \cos^{-1}\gamma$, and for $\gamma > 1$, this means that $\theta = i\cosh^{-1}\gamma = i\ln\left(\gamma + \sqrt{\gamma^2 - 1}\right)$. And

$\theta = \sin^{-1}(i\beta\gamma) = i\sinh^{-1}(\beta\gamma) = i\ln\left(\beta\gamma + \sqrt{\beta^2\gamma^2+1}\right)$. Either of these expressions reduces to $\theta = i\ln[\gamma(1+\beta)]$. Perhaps a yet more convenient way of expressing this is

$$\theta = i\tanh^{-1}\beta = \tfrac{1}{2}i\ln\left(\frac{1+\beta}{1-\beta}\right). \qquad 15.7.7$$

For example, if $\beta = 0.8$, $\theta = 1.0986i$, which might be written (not necessarily particularly usefully) as $i \times 62^{\circ}\ 57'$.

At this stage, you are probably thinking that you much prefer the version of equation 15.7.5, in which all quantities are real, and the expression for the interval between two events is $[(\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2 - (\Delta x_0)^2]^{1/2}$. The minus sign in the expression is a small price to pay for the realness of all quantities. Equation 15.7.5 can be written

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}, \qquad 15.7.8$$

where $\cosh\phi = \gamma,\quad \sinh\phi = \beta\gamma,\quad \tanh\phi = \beta$. On the face of it, this looks much simpler. No messing around with imaginary angles. Yet this formulation is not without its own set of difficulties. For example, neither the matrix of equation 15.7.5 nor the matrix of equation 15.7.8 is orthogonal. You cannot invert the equation to find x and t in terms of x' and t' merely by interchanging the primed and unprimed symbols and interchanging the rows and columns. The converse of equation 15.7.8 is in fact

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix}, \qquad 15.7.9$$

which can also (understandably!) be written

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \cosh(-\phi) & \sinh(-\phi) \\ \sinh(-\phi) & \cosh(-\phi) \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix}, \qquad 15.7.10$$

which demands as much skill in handling hyperbolic functions as the other formulation did in handling complex numbers. A further problem is that the formulation 15.7.5 does not allow the analogy between the Lorenz transformation and the rotation of axes. You take your choice.

It may be noticed that the determinants of the matrices of equations 15.7.5 and 15.7.8 are each unity, and it may therefore be thought that each matrix is orthogonal and that its reciprocal is its transpose. But this is not the case, for the condition that the determinant is unity is not a sufficient condition for a matrix to be orthogonal. The necessary tests are summarized in Celestial Mechanics, Section 3.7, and it will be found that several of the conditions are not satisfied.

15.8 *Timelike and Spacelike 4-Vectors*

I am going to refer some events to a coordinate system whose origin is here and now and which is moving at the same velocity as you happen to be moving. In other words, you are sitting at the origin of the coordinate system, and you are stationary with respect to it. Let us suppose that an event A occurs at the following coordinates referred to this reference frame, in which the distances x_1 , y_1 , z_1 are expressed in light-years (lyr) the time t_1 is expressed in years (yr).

$$x_1 = 2 \qquad y_1 = 3 \qquad z_1 = 7 \qquad t_1 = -1$$

A "light-year" is a unit of distance used when describing astronomical distances to the layperson, and it is also useful in describing some aspects of relativity theory. It is the distance travelled by light in a year, and is approximately 9.46×10^{15} m or 0.307 parsec (pc). Event A, then, occurred a year ago at a distance of $\sqrt{62} = 7.87$ lyr, when referred to this reference frame. Note that, if referred to a reference frame that coincides with this one at $t = 0$, but is moving with respect to it, all four coordinates might be different, and the distance $\sqrt{x^2 + y^2 + z^2}$ and the time of occurrence would be different, but, according to the way in which we have defined space and time by the Lorentz transformation, the quantity $\sqrt{x^2 + y^2 + z^2 - c^2t^2}$ would be the same.

Imagine now a second event, B, which occurs at the following coordinates:

$$x_2 = 5 \qquad y_2 = 8 \qquad z_2 = 10 \qquad t_2 = +2$$

That is to say, when referred to the same reference frame, it will occur in two years' time at a distance of $\sqrt{189} = 13.75$ lyr.

The 4-vector $\mathbf{s} = \mathbf{B} - \mathbf{A}$ connects these two events, and the magnitude s of $\mathbf{s}$ is the *interval* between the two events. Note that the *distance* between the two events, *when referred to our reference frame*, is $\sqrt{(5-2)^2 + (8-3)^2 + (10-7)^2} = 6.56$ lyr. The *interval* between the two events is $\sqrt{(5-2)^2 + (8-3)^2 + (10-7)^2 - (2+1)^2} = 5.83$ lyr, and this is independent of the velocity of the reference frame. That is, if we "rotate" the reference frame, it obviously makes no difference to the *interval* between the two events, which is *invariant*.

As another example, consider two events A and B whose coordinates are

$$x_1 = 2 \qquad y_1 = 5 \qquad z_1 = 3 \qquad t_1 = -2$$

$$x_2 = 3 \qquad y_2 = 7 \qquad z_2 = 4 \qquad t_2 = +6$$

with distances, as before, expressed in lyr, and times in yr. Calculate the interval between these two events – i.e. the magnitude of the 4-vector connecting them. If you carry out this calculation, you will find that $s^2 = -58$, so that the interval s is *imaginary* and equal to $7.62i$.

So we see that some pairs of events are connected by a 4-vector whose magnitude is real, and other pairs are connected by a 4-vector whose magnitude is imaginary. There are differences in character between real and imaginary intervals, but, in order to strip away distractions, I am going to consider events for which $y = z = 0$. We can now concentrate on the essentials without being distracted by unimportant details.

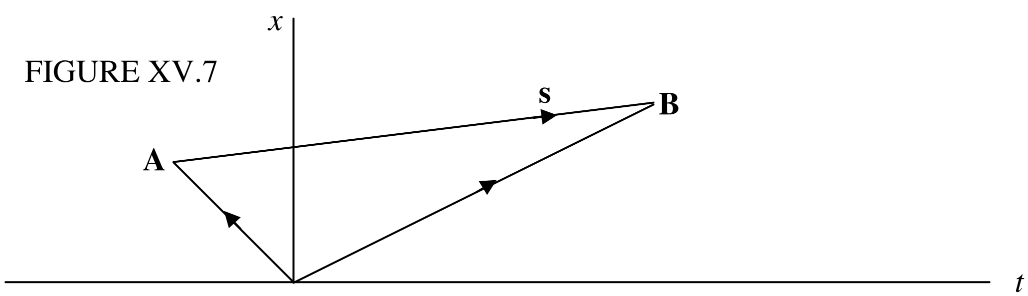
Let us therefore consider two events A and B whose coordinates are

$$x_1 = 2 \text{ lyr} \qquad t_1 = -2 \text{ yr}$$

$$x_2 = 3 \text{ lyr} \qquad t_2 = +6 \text{ yr}$$

These events and the 4-vector connecting them are shown in figure XV.7.Event A happened two years ago (referred to our reference frame); event B will occur (also referred to our reference frame) in six years' time. The square of the *interval* between the two events (which is invariant) is -63 lyr^2, and the interval is imaginary. If someone wanted to experience both events, he would have to travel only 1 lyr (referred to our reference frame), and he could take his time, for he would have eight years (referred to our reference frame) in which to make the journey to get to event B in time. He couldn't totally dawdle, however; he would have to travel at a speed of at least $\frac{1}{8}$ times the speed of light, but that's not extremely fast for anyone well versed in relativity.

FIGURE XV.7



Let's look at it another way. Let's suppose that event A is the *cause* of event B. This means that some agent must be capable of conveying some information from A to B at a speed at least equal to $\frac{1}{8}$ times the speed of light. That may present some technical problems, but it presents no problems to our imagination.

You'll notice that, in this case, the interval between the two events – i.e. the magnitude of the 4-vector connecting them – is *imaginary*. A 4-vector whose magnitude is *imaginary* is called a *timelike* 4-vector. There is quite a long time between events A and B, but not much distance.

Now consider two events A and B whose coordinates are

$$x_1 = 2 \text{ lyr} \qquad t_1 = -1 \text{ yr}$$

$$x_2 = 7 \text{ lyr} \qquad t_2 = +3 \text{ yr}$$

The square of the magnitude of the interval between these two events is +9 lyr^2, and the interval is real. A 4-vector whose magnitude is *real* is called a *spacelike* 4-vector. It is shown in figure XV.8.

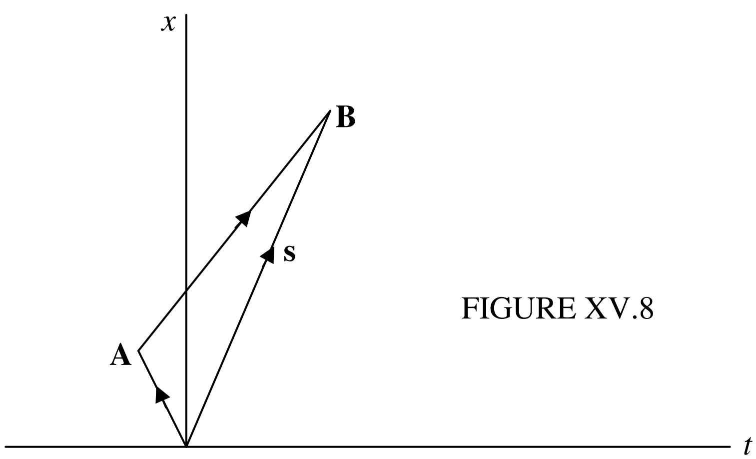


FIGURE XV.8

Perhaps I could now ask how fast you would have to travel if you wanted to experience both events. They are quite a long way apart, and you haven't much time to get from one to the other. Or, if event A is the *cause* of event B, how fast would an information-carrying agent have to move to convey the necessary information from A in order to instigate event B? Maybe you have already worked it out, but I'm not going to ask the question, because in a later section we'll find that *two events A and B cannot be mutually causally connected if the interval between them is real.* Note that I have said "mutually"; this means that A cannot cause B, and B cannot cause A. A and B must be quite independent events; there simply is too much space in the interval between them for one to be the cause of the other. It does not mean that the two events cannot have a common cause. Thus, figure XV.9 shows two events A and B with a spacelike interval between them (very steep) and a third event C such the intervals CA and CB (very shallow) are timelike. C could easily be the cause of both A and B; that is, A and B could have a common cause. But there can be no *mutual* causal connection between A and B. (It might be noted parenthetically that Charles Dickens temporarily nodded when he chose the title of his novel *Our Mutual Friend*. He really meant our common friend. C was a friend common to A and to B. A and B were friends mutually to each other.)

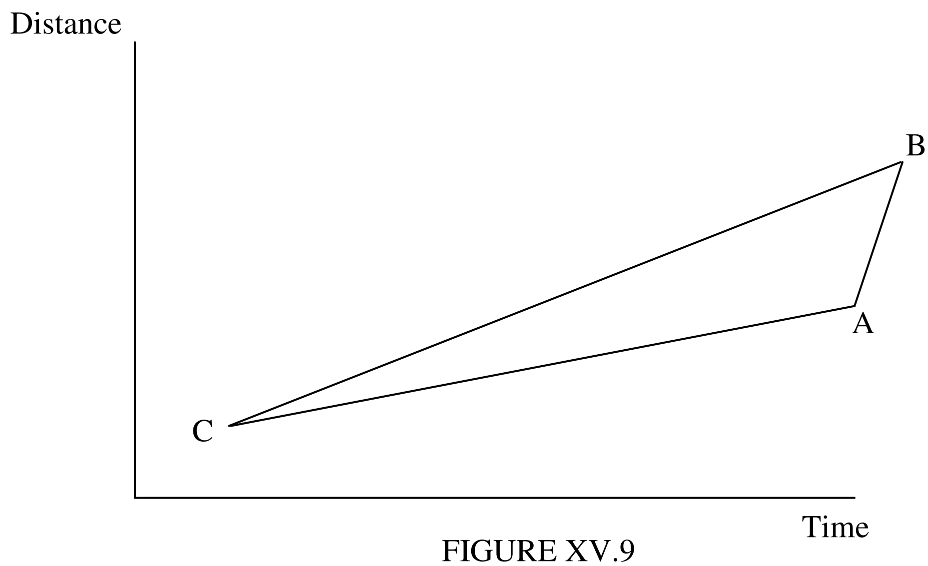


FIGURE XV.9

Exercise. The distance of the Sun from Earth is 1.496×10^{11} m. The speed of light is 2.998×10^{8} m s^{-1}. How long does it take for a photon to reach Earth from the Sun? Event A: A photon leaves the Sun on its way to Earth. Event B: The photon arrives at Earth. What is the interval (i.e. s in 4-space) between these two events?

15.9 *The FitzGerald-Lorentz Contraction*

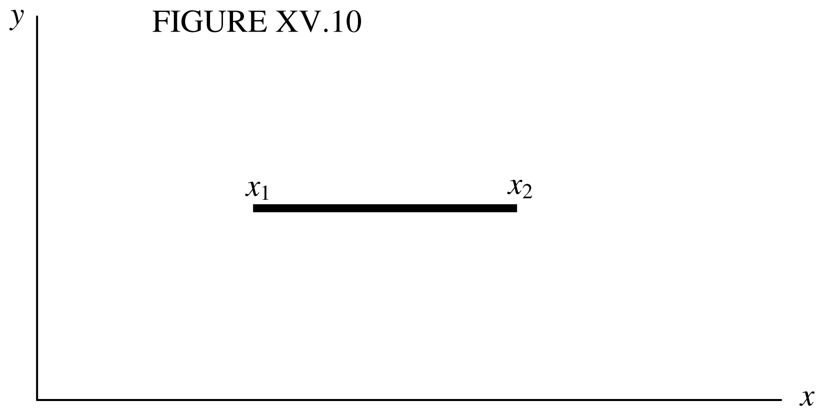
This is sometimes described in words something like the following:

If a measuring-rod is moving with respect to a "stationary" observer, it "appears" to be shorter than it "really" is.

This is not a very precise statement, and the words that I have placed in inverted commas call for some clarification.

We have seen that, while the interval between two events is invariant between reference frames, the distance between two points (and hence the length of a rod) depends on the coordinate frame to which the points are referred. Let us now define what we mean by the *length* of a rod. Figure XV.10 shows a reference frame, and a rod lying parallel to the x-axis. For the moment I am not specifying whether the rod is moving with respect to the reference frame, or whether it is stationary.

FIGURE XV.10



Let us suppose that the x-coordinate of the left-hand end of the rod is x_1, and that, *at the same time referred to this reference frame*, the x-coordinate of the right-hand end is x_2. The length l of the rod is defined as $l = x_2 - x_1$. That could scarcely be a simpler statement – but note the little phrase "at the same time referred to this reference frame". That simple phrase is important.

Now let's look at the FitzGerald-Lorentz contraction. See figure XV.11.

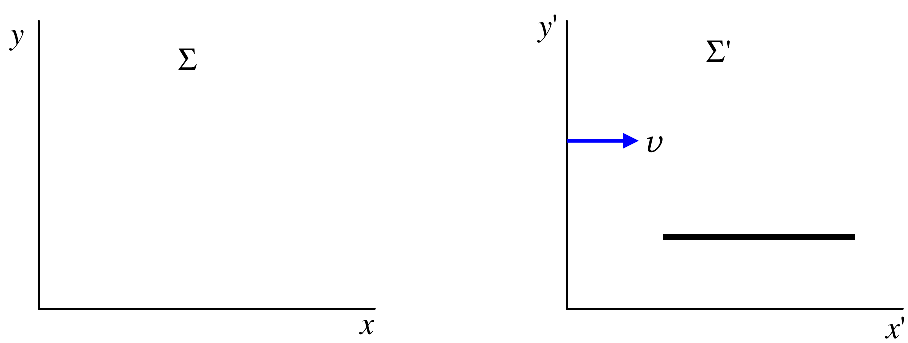


FIGURE XV.11

There are two reference frames, Σ and Σ'. The frame Σ' is moving to the right with respect to Σ with speed υ. A rod *is at rest with respect to the frame* Σ', and is therefore moving to the right with respect to Σ at speed υ.

In my younger days I often used to travel by train, and I still like to think of railway trains whenever I discuss relativity. Modern students usually like to think of spacecraft, presumably because they are more accustomed to this mode of travel. In the very early days of railways, it was customary for the stationmaster to wear top hat and tails. Those days are long gone, but, when thinking about the FitzGerald-Lorentz contraction, I like to think of Σ as being a railway station in which there resides a stationmaster in top hat and tails, while Σ' is a railway train.

The length of the rod, referred to the frame Σ', is $l' = x'_2 - x'_1$, in what I hope is obvious notation, and of course these two coordinates are determined at the same time referred to Σ'.

The length of the rod *referred to a frame in which it is at rest* is called its *proper length.* Thus l' is the proper length of the rod.

Now it should be noted that, according to the way in which we have defined distance and time by means of the Lorentz transformation, although x'_2 and x'_1 are measured simultaneously with respect to Σ', these two events (the determination of the coordinates of the two ends of the rod) are not simultaneous when referred to the frame Σ (a point to which we shall return in a later section dealing with simultaneity). The length of the rod referred to the frame Σ is given by $l = x_2 - x_1$, where these two coordinates are to be determined at the same time when referred to Σ. Now equation 15.5.16 tells us that $x_2 = x'_2/\gamma + \upsilon t$ *and* $x_1 = x'_1/\gamma + \upsilon t.$ (Readers should note this derivation very carefully, for it is easy to go wrong. In particular, be very clear what is meant in these two equations by the symbol *t*. It is the single instant of time, referred to Σ, when the coordinates of the two ends are determined simultaneously with respect to Σ.) From these we reach the result:

$$l = l'/\gamma . \qquad 15.9.1$$

This is the FitzGerald-Lorentz contraction.

It is sometimes described thus: A railway train of proper length 100 yards is moving past a railway station at 95% of the speed of light ($\gamma = 3.2026.$) To the stationmaster the train “appears” to be of length 31.22 yards; or the stationmaster “thinks” the length of the train is 31.22 yards; or, “according to” the stationmaster the length of the train is 31.22 yards. This gives a false impression, as though the stationmaster is under some sort of misapprehension concerning the length of the train, or as if he is labouring under some sort of illusion, and it introduces some sort of unnecessary “mystery” into what is nothing more than simple algebra. In fact what the stationmaster “thinks” or “asserts” is entirely irrelevant. Two correct statements are: 1. The length of the train, referred to a reference frame in which it is at rest – i.e. the proper length of the train – is 100 yards. 2. The length of the train when referred to a frame with respect to which it is moving at a speed of 0.95*c* is 31.22 yards. And that is all there is to it. Any phrase such as “this observer thinks that” or “according to this observer” should always be interpreted in this manner. It is not a matter of what an observer “thinks”. It is a matter of which frame a measurement is referred to. Nothing more, nothing less.

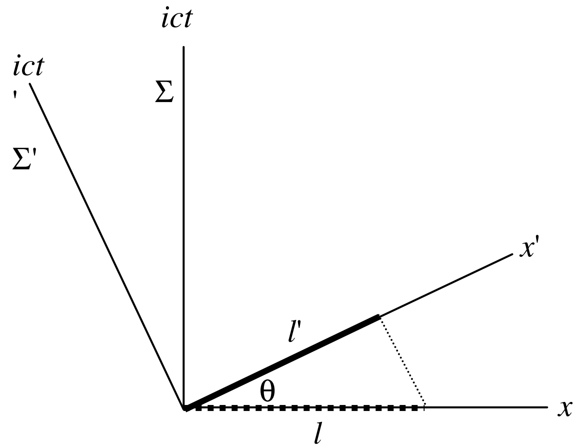


FIGURE XV.12

It is possible to describe the Lorentz-FitzGerald contraction by interpreting the Lorentz transformations as a rotation in 4-space. Whether it is helpful to do so only you can decide. Thus figure XV.12 shows Σ and Σ' related by a rotation in the manner described in section 15.7. The thick continuous line shows a rod oriented so that its two ends are drawn at the same time with respect to Σ'. Its length is, referred to Σ', l', and this is its proper length. The thick dotted line shows the two ends at the same time with respect to Σ. Its length referred to Σ is $l = l'/\cos\theta$. And, since $\cos\theta = \gamma$, which is greater than 1,, this means that, in spite of appearances in the figure, $l < l'$. The figure is deceptive because, as discussed in section 15.7, θ is imaginary. As I say, only you can decide whether this way of looking at the contraction is helpful or merely confusing. It is, however, at least worth looking at, because I shall be using this concept of rotation in a forthcoming section on simultaneity and order of events. Illustrating the Lorentz transformations as a rotation like this is called a *Minkowski diagram.*

15.10 *Time Dilation*

We imagine the same railway train Σ' and the same railway station Σ as in the previous section except that, rather than measuring a length referred to the two reference frames, we measure the time interval between two events. We'll suppose that a passenger in the railway train Σ' claps his hands twice. These are two events which take place *at the same place when referred to this reference frame* Σ'. Let the instants of time when the two events occur, referred to Σ', be t'_1 and t'_2. The time interval T' is defined as $t'_2 - t'_1$. But the Lorentz transformation is $t = \gamma(t' + vx'/c^2)$, and so the time interval when referred to Σ is

$$T = \gamma T'. \qquad 15.10.1$$

This is the *dilation of time*. The situation is illustrated by a Minkowski diagram in figure XV.13. While it is clear from the figure that $T = T'\cos\theta$ and therefore that $T = \gamma T'$, it is not so clear from the figure that this means that T is greater than T' – because cos θ > 1 and θ is imaginary.

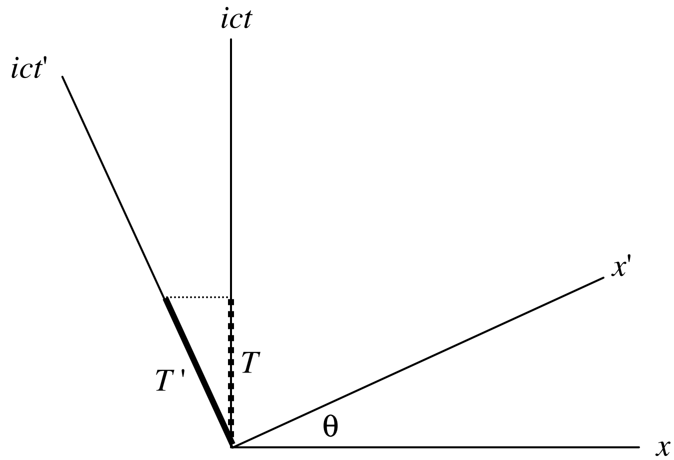


FIGURE XV.13

Thus, let us suppose that a passenger on the train holds a 1-metre measuring rod (its length in the direction of motion of the train) and he claps his hands at an interval of one second apart. Let's suppose that the train is moving at 98% of the speed of light (γ = 5.025). In that case the stationmaster thinks that the length of the rod is only 19.9 cm and that the time interval between the claps is 5.025 seconds.

I deliberately did not word that last sentence very well. It is not a matter of what the stationmaster or anyone else "thinks" or "asserts". It is not a matter that the stationmaster is somehow deceived into erroneously believing that the rod is 19.9 cm long and the claps 5.025 seconds apart, whereas they are "really" 1 metre long and 1 second apart. It is a matter of how length and time are defined (by subtracting two space coordinates determined at the same time, or two time coordinates at the same place) and how space-time coordinates are defined by means of the Lorentz transformations. The length *is* 19.9 cm, and the time interval *is* 5.025 seconds when referred to the frame Σ. It is true that the *proper length* and the *proper time interval* are the length and the time interval referred to a frame in which the rod and the clapper are at rest. In that sense one could loosely say that they are "really" 1 metre long and 1 second apart. But the Lorentz contraction and the time dilation are not determined by what the stationmaster or anyone else "thinks".

Another way of looking at it is this. The interval s between two events is clearly independent of the orientation any reference frames, and is the same when referred to two reference frames that may be inclined to each other. But the components of the vector joining two events, or their projections on to the time axis or a space axis are not at all expected to be equal.

By the way, in section 15.3 I urged you to write a computer or calculator programme for the instant conversion between the several factors commonly encountered in relativity. I still urge it. As soon as I typed that the train was travelling at 98% of the speed of light, I was instantly able to generate γ. You need to be able to do that, too.

15.11 *The Twins Paradox*

During the late 1950s and early 1960s there was great controversy over a problem known as the "Twins Paradox". The controversy was not confined to within scientific circles, but was argued, by scientists and others, in the newspapers, magazines and many serious journals. It goes something like this:

There are two 20-year-old twins, Albert and Betty. Albert is a sedentary type who likes nothing better than to stay at home tending the family vineyards. His twin sister Betty is a more adventurous type, and has trained to become an astronaut. On their twentieth birthday, Betty waves a cheery *au revoir* to her brother and takes off on what she intends to be a brief spaceflight, at which she travels at 99.98 % of the speed of light ($\gamma = 50$). After six months by her calendar she turns back and on her 21st birthday she arrives back home to greet her brother, only to find that he is now old and sere and has laboured, by his calendar for 50 years and is now an aged man of 71 years. If we accept what we have derived in the previous section about the dilation of time, there would seem to be no particular problem with that. It has even been argued that travel between the stars may not be an impossibility. Whereas to an Earthbound observer it may take many decades for a spacecraft to travel to a star and back, for the astronauts on board much less time has elapsed.

And yet a paradox was pointed out. According to the principles of the relativity of motion, it was argued, one could refer everything to Betty's reference frame, and from that point of view one could regard Betty as being the stationary twin and Albert as the one who travelled off into the distance and returned later. Thus, it could be argued, it would be Albert who had aged only one year, while Betty would have aged fifty years. Thus we have a *paradox*, which is a problem which apparently gives rise to opposite conclusions depending on how it is argued. And the only way that the paradox could be resolved was to suppose that both twins were the same age when they were re-united.

A second argument in favour of this interpretation that the twins were the same age when re-united points out that dilation of time arises because two events that may occur in the same place when referred to one reference frame do not occur in the same place when referred to another. But in this case, the two events (Betty's departure and re-arrival) occur at the same place when referred to both reference frames.

The argument over this point raged quite furiously for some years, and a particularly plausible tool that was used was something referred to as the "*k*-calculus" – an argument that is, however, fatally flawed because the "rules" of the *k*-calculus inherently incorporate the desired conclusion. Two of the principal leaders of the very public scientific debate were Professors Fred Hoyle and Herbert Dingle, and this inspired the following letter to a weekly magazine, *The Listener*, in 1960:

Sir:

The ears of a Hoyle may tingle;
The blood of a Hoyle may boil
When Hoyle pours hot oil upon Dingle,
And Dingle cold water on Hoyle.

But the dust of the wrangle will settle.
Old stars will look down on new soil.
The pot will lie down with the kettle,
And Dingle will mingle with Hoyle.

So what are you, the reader, expected to believe? Let us say this: If you are a student who has examinations to pass, or if you are an untenured professor who has to hold on to a job, be in no doubt whatever: The original conclusion is the canonically-accepted correct conclusion, namely that Albert has aged 50 years while his astronaut sister has aged but one. This is now firmly accepted truth. Indeed it has even been claimed that it has been "proved" experimentally by a scientist who took a clock on commercial airline flights around the world, and compared it on his return with a stay-at-home clock. For myself I have neither examinations to pass nor, alas, a job to hold on to, so I am not bound to believe one thing or the other, and I elect to hold my peace.

I do say this, however – that what anyone "believes" is not an essential point. It is not a matter of what Albert or Betty or Hoyle or Dingle or your professor or your employer "believes". The real question is this: What is it that is predicted by the special theory of relativity? From this point of view it does not matter whether the theory of relativity is "true" or not, or whether it represents a correct description of the real physical world. Starting from the basic precepts of relativity, whether "true" or not, it must be only a matter of algebra (and simple algebra at that) to decide what is predicted by relativity.

A difficulty with this is that it is not, strictly speaking, a problem in special relativity, for special relativity deals with transformations between reference frames that are in uniform motion relative to one another. It is pointed out that Albert and Betty are not in uniform motion relative to one another, since one or the other of them has to change the direction of motion – i.e. has to accelerate. It could still be argued that, since motion is relative, one can regard either Albert or Betty as the one who accelerates – but the response to this is that only *uniform* motion is relative. Thus there is no symmetry between Albert and Betty. Betty either accelerates or experiences a gravitational field (depending on whether

her experience is referred to Albert's or her own reference frame). And, since there is no symmetry, there is no paradox. This argument, however, admits that the age difference between Albert and Betty on Betty's return is not an effect of special relativity, but of general relativity, and is an effect caused by the acceleration (or gravitational field) experienced by Betty.

If this is so, there are some severe difficulties is describing the effect under general relativity. For example, whether the general theory allows for an instantaneous change in direction by Betty (and infinite deceleration), or whether the final result depends on how she decelerates – at what rate and for how long – must be determined by those who would tackle this problem. Further, the alleged age difference is supposed to depend upon the time during which Betty has been travelling and the length of her journey – yet the portion of her journey during which she is accelerating or decelerating can be made arbitrarily short compared with the time during which she is travelling at constant speed. If the effect were to occur solely during the time when she was accelerating or decelerating, then the total length and duration of the constant speed part of her journey should not affect the age difference at all.

Since this chapter deals only with special relativity, and this is evidently not a problem restricted to special relativity, I leave the problem, as originally stated, here, without resolution, for readers to argue over as they will

15.12 *A, B and C*

A, B and C were three characters in the Canadian humorist Stephen Leacock's essay on *The Human Element in Mathematics*. "A, B and C are employed to dig a ditch. A can dig as much in one hour as B can dig in two..."

We can ask A, B and C to come to our aid in a *modified version* of the twins' problem, for we can arrange all three of them to be moving with constant velocities relative to each other. It goes like this (figure XV.14):

FIGURE XV.14

The scenario is probably obvious from the figure. There are three events:

1. B passes A
2. B meets C
3. C passes A

At event 1, B and A synchronize their watches so that each reads zero. At event 2, C sets his watch so that it reads the same as B's. At event 3, C and A compare watches. I shall leave the reader to cogitate over this. The only thing I shall point out is that this problem differs from the problem described as the Twins Paradox in two ways. In the first place, unlike in the Twins Paradox, all three characters, A, B and C are moving at constant velocities with respect to each other. Also, the first and third events occur at the same place relative to A but at different places referred to B or to C. In the twin paradox problem, the two events occur at the same place relative to both frames.

15.13 *Simultaneity*

If the time interval referred to one reference frame can be different when referred to another reference frame (and since time interval is merely one component of a four-vector, the magnitude of the component surely depends on the orientation in four space of the four axes) this raises the possibility that there might be a time interval of zero relative to one frame (i.e. two events are simultaneous) but are not simultaneous relative to another. This is indeed the case, provided that the two events do not occur in the same place as well as at the same time. Look at figure XV.15.

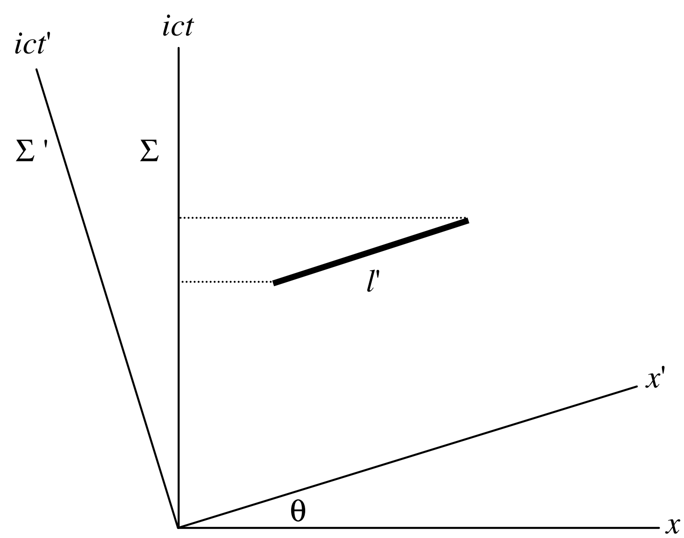


FIGURE XV.15

I have drawn two reference frames at an (imaginary) angle θ to each other. Think of Σ as the railway station and of Σ' as the railway train, and that the speed of the railway train is $c \tan\theta$. (You may have to go back to section 15.3 or 15.7 to recall the relation of θ to the speed.) The thick line represents the interval between two events that are simultaneous when referred to Σ', but are separated in space (one occurs near the front of the train; the other occurs near the rear). (Note also in this text that I am using the phrase "time interval" to denote the time-component of the "interval". For two simultaneous events, the *time interval* is zero, and the *interval* is then merely the distance between the two events.)

While the thick line has zero component along the *ict*' axis, its component along the *ict* axis is $l'\sin\theta$. That is, $ic(t_2 - t_1) = l'\sin\theta = l' \times i\beta\gamma$.

Hence:

$$t_2 - t_1 = \frac{\beta\gamma l'}{c}. \tag{15.13.1}$$

For example, if the events took place simultaneously 100,000 km apart in the train (it is a long train) and if the train were travelling at 95% of the speed of light ($\gamma = 3.203$; it is a fast train), the two events would be separated when referred to the railway station by 1.01 seconds. The event near the rear of the train occurred first.

15.14 *Order of Events, Causality and the Transmission of Information*

Maybe it is even possible that if one event precedes another in one reference frame, in another reference frame the other precedes the one. In other words, the order of occurrence of events may be different in two frames. This indeed can be the case, and Minkowski diagrams (figure XV.16) can help us to see why and in what circumstances.

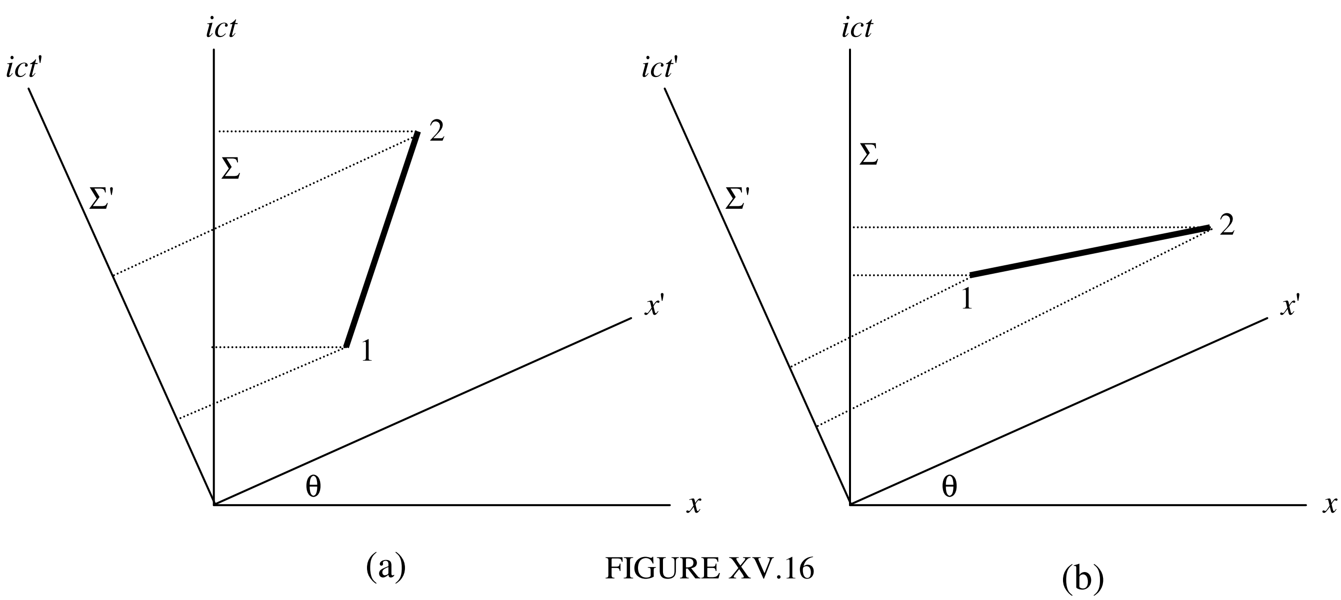


FIGURE XV.16

In part (a), of the two events 1 and 2, 1 occurs before 2 in either Σ or Σ'. (from this point on I shall use a short phrase such as "in Σ" rather than the more cumbersome "when referred to the reference frame Σ". But in part (b), event 1 occurs before event 2 in Σ, but after event 2 in Σ'. One can see that there is reversal of order of events if the slope of the line joining to two events is less than the angle θ. The angle θ, it may be recalled, is an imaginary angle such than $\tan\theta = i\beta = i\upsilon/c$, where υ is the relative speed of the two frames. In figure XV.17, for simplicity I am going to suppose that event 1 occurs at the origin of both frames, and that event 2 occurs at coordinates (υt , ict) in Σ. The condition for no reversal of events is then evidently

$$\frac{ict}{\upsilon t} \geq \tan\theta = i\beta = \frac{i\upsilon}{c};$$

or

$$\upsilon \leq c. \qquad 15.14.1$$

Now suppose that events 1 and 2 are *causally connected* in the sense that event 1 is the cause of event 2. For this to be the case, some signal carrying information must travel from 1 to 2. However, if event 1 is the cause of event 2, event 1 must precede event 2 in all reference frames. Thus it follows that *no signal carrying information that could cause an event to occur can travel faster than the speed of light.*

This means, in effect, that neither mass nor energy can be transmitted faster than the speed of light. That is not quite the same thing as saying that "nothing" can be transmitted faster than the speed of light. For example a Moiré pattern formed by two combs with slightly different tooth spacings can move faster than light if one of the combs is moved relative to the other; but then I suppose it has to be admitted that in that case "nothing" is actually being transmitted – and certainly nothing that can transmit information or that can cause an event. An almost identical example would be the modulation envelope of the sum of two waves of slightly different frequencies. A well-known example from wave mechanics is that of the wave representation of a moving particle. The wave group (which is the integral of a continuous distribution of wavelengths whose extent is governed by Heisenberg's principle) moves with the particle at a sub-luminal speed, but there is nothing to prevent the wavelets within the group moving through the group at any speed. These wavelets may start at the beginning of the group and rapidly move through the group and extinguish themselves at the end. No "information" is transmitted from A to B at a speed any faster than the particle itself is moving.

15.15 *Derivatives*

We'll pause here and establish a few derivatives just for reference and in case we need them later.

We recall that the Lorentz relations are

$$x = \gamma(x' + \upsilon t') \qquad 15.15.1$$

and $$t = \gamma\left(t' + \frac{\beta x'}{c}\right).$$ 15.15.2

From these we immediately find that

$$\left(\frac{\partial x}{\partial x'}\right)_{t'} = \gamma; \quad \left(\frac{\partial x}{\partial t'}\right)_{x'} = \gamma\upsilon; \quad \left(\frac{\partial t}{\partial x'}\right)_{t'} = \frac{\beta\gamma}{c}; \quad \left(\frac{\partial t}{\partial t'}\right)_{x'} = \gamma.$$ 15.15.3a,b,c,d

We shall need these in future sections.

Caution: It is not impossible to make a mistake with some of these derivatives if one allows one's attention to wander. For example, one might suppose that, since $\partial x/\partial x' = \gamma$, then "obviously" $\partial x'/\partial x = 1/\gamma$ - and indeed this is correct if t' is being held constant. However, we have to be sure that this is really what we want. The difficulty is likely to arise if, when writing a partial derivative, we neglect to specify what variables are being held constant, and no great harm would be done by insisting that these always be specified when writing a partial derivative. If you want the *inverses* rather than the *reciprocals* of equations 15.15.3a,b,c,d, the rule, as ever, is: Interchange the primed and unprimed symbols and change the sign of υ or β. For example, the reciprocal of $\left(\frac{\partial x}{\partial x'}\right)_{t'}$ is $\left(\frac{\partial x'}{\partial x}\right)_{t'}$, while its inverse is $\left(\frac{\partial x'}{\partial x}\right)_{t}$. For completeness, and reference, then, I write down all the possibilities:

$$\left(\frac{\partial x'}{\partial x}\right)_{t'} = 1/\gamma; \quad \left(\frac{\partial t'}{\partial x}\right)_{x'} = 1/\gamma\upsilon; \quad \left(\frac{\partial x'}{\partial t}\right)_{t'} = \frac{c}{\beta\gamma}; \quad \left(\frac{\partial t'}{\partial t}\right)_{x'} = 1/\gamma.$$ 15.15.3e,f,g,h

$$\left(\frac{\partial x'}{\partial x}\right)_{t} = \gamma; \quad \left(\frac{\partial x'}{\partial t}\right)_{x} = -\gamma\upsilon; \quad \left(\frac{\partial t'}{\partial x}\right)_{t} = -\frac{\beta\gamma}{c}; \quad \left(\frac{\partial t'}{\partial t}\right)_{x} = \gamma.$$ 15.15.3i.j.k.l

$$\left(\frac{\partial x}{\partial x'}\right)_{t} = 1/\gamma; \quad \left(\frac{\partial t}{\partial x'}\right)_{x} = -1/\gamma\upsilon; \quad \left(\frac{\partial x}{\partial t'}\right)_{t} = -\frac{c}{\beta\gamma}; \quad \left(\frac{\partial t}{\partial t'}\right)_{x} = 1/\gamma.$$ 15.15.3m,n,o,p

Now let's suppose that $\psi = \psi(x,t)$, where x and t are in turn functions (equations15.15.1 and 15.1.5.2) of x' and t'. Then

$$\frac{\partial\psi}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial\psi}{\partial x} + \frac{\partial t}{\partial x'}\frac{\partial\psi}{\partial t} = \gamma\frac{\partial\psi}{\partial x} + \frac{\beta\gamma}{c}\frac{\partial\psi}{\partial t}$$ 15.15.4

and $$\frac{\partial\psi}{\partial t'} = \frac{\partial x}{\partial t'}\frac{\partial\psi}{\partial x} + \frac{\partial t}{\partial t'}\frac{\partial\psi}{\partial t} = \gamma\upsilon\frac{\partial\psi}{\partial x} + \gamma\frac{\partial\psi}{\partial t}\cdot$$ 15.15.5

The reader will doubtless notice that I have here ignored my own advice and I have not indicated which variables are to be held constant. It would be worth spending a moment here thinking about this.

We can write equations 15.15.4 and 15.15.5 as equivalent operators:

$$\frac{\partial}{\partial x'} \equiv \gamma\left(\frac{\partial}{\partial x} + \frac{\beta}{c}\frac{\partial}{\partial t}\right) \qquad 15.15.6$$

and

$$\frac{\partial}{\partial t'} \equiv \gamma\left(v\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right). \qquad 15.15.7$$

We can also, if we wish, find the second derivatives. Thus

$$\frac{\partial^2\psi}{\partial x'^2} = \frac{\partial}{\partial x'}\frac{\partial\psi}{\partial x'} = \gamma^2\left(\frac{\partial}{\partial x} + \frac{\beta}{c}\frac{\partial}{\partial t}\right)\left(\frac{\partial\psi}{\partial x} + \frac{\beta}{c}\frac{\partial\psi}{\partial t}\right), \qquad 15.15.8$$

from which we find

$$\frac{\partial^2}{\partial x'^2} \equiv \gamma^2\left(\frac{\partial^2}{\partial x^2} + \frac{2\beta}{c}\frac{\partial^2}{\partial x\partial t} + \frac{\beta^2}{c^2}\frac{\partial^2}{\partial t^2}\right). \qquad 15.15.9$$

In a similar manner we obtain

$$\frac{\partial^2}{\partial x'\partial t'} \equiv \gamma^2\left(v\frac{\partial^2}{\partial x^2} + \left(1+\beta^2\right)\frac{\partial^2}{\partial x\partial t} + \frac{\beta}{c}\frac{\partial^2}{\partial t^2}\right) \qquad 15.15.10$$

and

$$\frac{\partial^2}{\partial t'^2} \equiv \gamma^2\left(v^2\frac{\partial^2}{\partial x^2} + 2v\frac{\partial^2}{\partial x\partial t} + \frac{\partial^2}{\partial t^2}\right). \qquad 15.15.11$$

The inverses of all of these relations are to be found by interchanging the primed and unprimed coordinates and changing the signs of v and β.

15.16 *Addition of velocities*

A railway train trundles towards the east at speed v_1, and a passenger strolls towards the front at speed v_2. What is the speed of the passenger relative to the railway station? We might at first be tempted to reply: "Why, $v_1 + v_2$, of course." In this section we shall show that the answer as predicted from the Lorentz transformations is a little less than this, and we shall develop a formula for calculating it. We have already discussed (in section 15.6) our answer to the objection that this defies common sense. We pointed out there that the answer (to the perfectly reasonable objection) that "at the speeds we are accustomed to we would hardly notice the difference" is not a satisfactory response. The reason that the resultant speed is a little less than $\upsilon_1 + \upsilon_2$ results from the way in which we have defined the Lorentz transformations between references frames and the way in which distances and time intervals are defined with reference to reference frames in uniform relative motion.

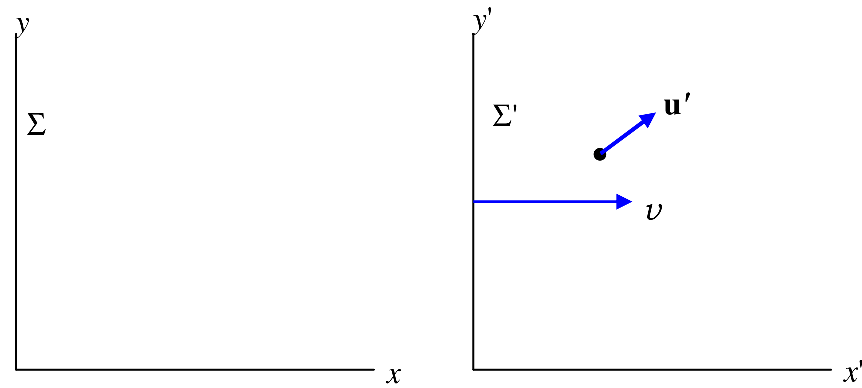


FIGURE XV.17

Figure XV.17 shows two references frames, Σ and Σ', the latter moving at speed υ with respect to the former. A particle is moving with velocity **u'** in Σ', with components $u'_{x'}$ and $u'_{y'}$. (" in Σ' " = "referred to the reference frame Σ' ".)

What is the velocity of the particle in Σ?

Let us start with the *x*-component.

We have:

$$u = \frac{dx}{dt} = \frac{\left(\frac{\partial x}{\partial x'}\right)_{t'} dx' + \left(\frac{\partial x}{\partial t'}\right)_{x'} dt'}{\left(\frac{\partial t}{\partial x'}\right)_{t'} dx' + \left(\frac{\partial t}{\partial t'}\right)_{x'} dt'} = \frac{\left(\frac{\partial x}{\partial x'}\right)_{t'} u' + \left(\frac{\partial x}{\partial t'}\right)_{x'}}{\left(\frac{\partial t}{\partial x'}\right)_{t'} u' + \left(\frac{\partial t}{\partial t'}\right)_{x'}} . \qquad 15.16.1$$

We take the derivatives from equations 15.15.3a-d, and, writing υ /c for β, we obtain

$$u_x = \frac{u'_{x'} + \upsilon}{1 + u'_{x'}\upsilon / c^2} . \qquad 15.16.2$$

The inverse is obtained by interchanging the primed and unprimed symbols and reversing the sign of υ.

The y-component is found in an exactly similar manner, and I leave its derivation to the reader. The result is

$$u_y = \frac{u'_{y'}}{\gamma(1 + u'_{x'}\upsilon / c^2)}. \qquad 15.16.3$$

Special cases:

I. If $u'_{x'} = u'$ and $u'_{y'} = 0$, then

$$u_x = \frac{u' + \upsilon}{1 + u'\upsilon / c^2} \quad \text{and} \quad u_y = 0. \qquad 15.16.4\text{a,b}$$

II. If $u'_{x'} = 0$ and $u'_{y'} = u'$, then

$$u_x = \upsilon \quad \textit{and} \quad u_y = u'/\gamma. \qquad 15.16.5\text{a,b}$$

Equation 15.16.4a as written is not easy to commit to memory, though it is rather easier if we write $\beta_1 = \upsilon / c$, $\beta_2 = u'/c$ and $\beta = u_x / c$. Then the equation becomes

$$\beta = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}. \qquad 15.16.6$$

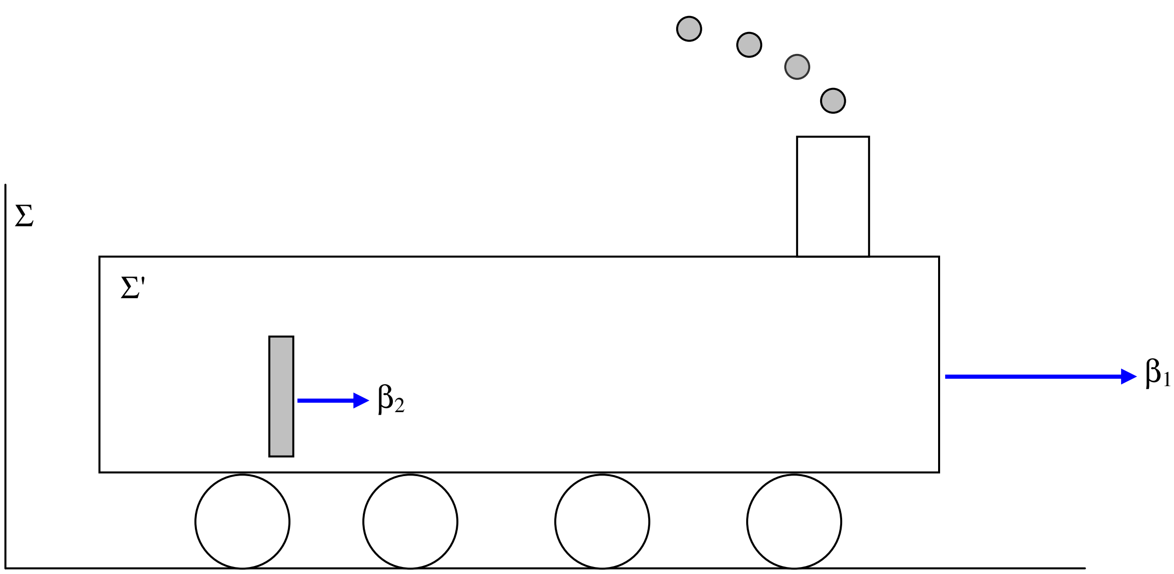


FIGURE XV.18

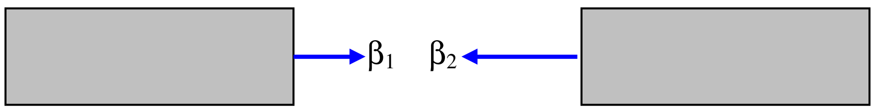


FIGURE XV.19

In figure XV.18, a train Σ' is trundling with speed β_1 (times the speed of light) towards the right, and a passenger is strolling towards the front at speed β_2. The speed β of the passenger relative to the station Σ is then given by equation 15.16.6. In figure XV.19, two trains, one moving at speed β_1 and the other moving at speed β_2, are moving towards each other. (If you prefer to think of protons rather than trains, that is fine.) Again, the relative speed β of one train relative to the other is given by equation 15.16.6.

Example. A train trundles to the right at 90% of the speed of light relative to Σ, and a passenger strolls to the right at 15% of the speed of light relative to Σ'. The speed of the passenger relative to Σ is 92.5% of the speed of light.

The relation between β_1 , β_2 and β is shown graphically in figure XV.20.

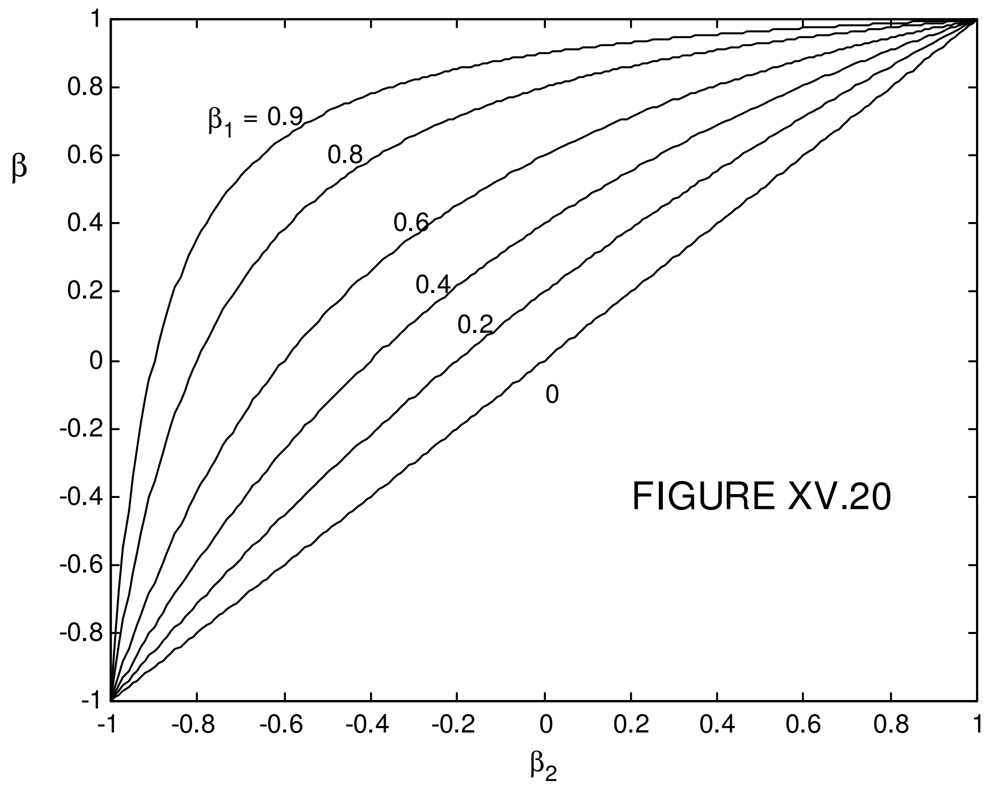


FIGURE XV.20

If I use the notation $\beta_1 \oplus \beta_2$ to mean "combining β_1 with β_2", I can write equation 15.16.6 as

$$\beta_1 \oplus \beta_2 = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2} . \qquad 15.16.7$$

You may notice the similarity of equation 15.16.6 $\beta = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}$ to the hyperbolic function identity

$$\tanh(\phi_1 + \phi_2) = \frac{\tanh\phi_1 + \tanh\phi_2}{1 + \tanh\phi_1 \tanh\phi_2} . \qquad 15.16.8$$

Thus I can represent the speed of an object by giving the value of ϕ, where

$$\beta = \tanh\phi \qquad 15.16.9$$

or

$$\phi = \tanh^{-1}\beta = \tfrac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right). \qquad 15.16.10$$

The factor ϕ combines simply as

$$\phi_1 \oplus \phi_2 = \phi_1 + \phi_2. \qquad 15.16.11$$

If you did what I suggested in section 15.3 and programmed your calculator or computer to convert instantly from one relativity factor to another, you now have a quick way of adding speeds.

Example. A train trundles to the right at 90% of the speed of light (ϕ_1 = 1.47222) relative to Σ, and a passenger strolls to the right at 15% of the speed of light (ϕ_2 = 0.15114) relative to Σ'. The speed of the passenger relative to Σ is ϕ = 1.62336, or 92.5% of the speed of light.

Example.

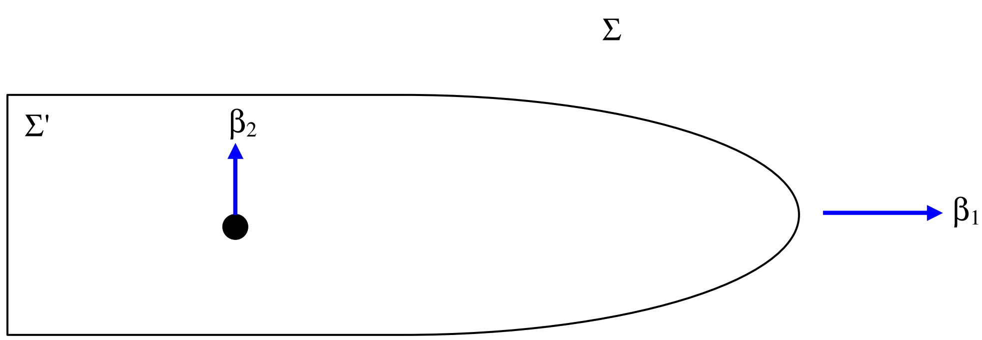


FIGURE XV.22

(Sorry – there is no figure XV.21.)

An ocean liner Σ' sails serenely eastwards at a speed $\beta_1 = 0.9c$ ($\gamma_1 = 2.29416$) relative to the ocean Σ. A passenger ambles athwartships at a speed $\beta_2 = 0.5c$ relative to the ship. What is the velocity of the passenger relative to the ocean?

The northerly component of her velocity is given by equation 15.16.5b, and is 0.21794c. Her easterly component is just 0.9c. Her velocity relative to the ocean is therefore 0.92601c in a direction 13° 37' north of east.

Exercise. Show that, if the speed of the ocean liner is β_1 and the athwartships speed of the passenger is β_2, the resultant speed β of the passenger relative to the ocean is given by

$$\beta^2 = \beta_1^2 + \beta_2^2 - \beta_1^2\beta_2^2 \qquad 15.16.12$$

and that her velocity makes and angle α with the velocity of the ship given by

$$\tan\alpha = \beta_2\sqrt{1-\beta_1^2}/\beta_1 . \qquad 15.16.13$$

Example.

A railway train Σ' of proper length L_0 = 100 yards thunders past a railway station Σ at such a speed that the stationmaster thinks its length is only 40 yards. (Correction: It is not a matter of what he "thinks". What I should have said is that the length of the train, referred to a reference frame Σ in which the stationmaster is at rest, is 40 yards.) A dachshund waddles along the corridor towards the front of the train. (A dachshund, or badger hound, is a cylindrical dog whose proper length is normally several times its diameter.) The proper length l_0 of the dachshund is 24 inches, but to a seated passenger, it appears to be... no, sorry, I mean that its length, referred to the reference frame Σ', is 15 inches. What is the length of the dachshund referred to the reference frame Σ in which the stationmaster is at rest?

We are told, in effect, that the speed of the train relative to the station is given by $\gamma_1 = 2.5$, and that the speed of the dachshund relative to the train is given by $\gamma_2 = 1.6$. So how do these two gammas combine to make the factor γ for the dachshund relative to the station?

There are several ways in which you could do this problem. One is to develop a general algebraic method of combining two gamma factors. Thus:

Exercise. Show that two gamma factors combine according to

$$\gamma_1 \oplus \gamma_2 = \gamma_1\gamma_2 + \sqrt{(\gamma_1^2 - 1)(\gamma_2^2 - 1)} . \qquad 15.16.14$$

I'll leave you to try that. The other way is to take advantage of the programme you wrote when you read section 15.3, by which you can instantaneously convert one relativity factor to another. Thus you instantly convert the gammas to phis.

Thus $\gamma_1 = 2.5 \Rightarrow \phi_1 = 1.56680$
and $\gamma_1 = 1.6 \Rightarrow \phi_1 = 1.04697$
$\therefore \qquad \phi = 2.61377 \Rightarrow \gamma = 6.86182.$

Is this what equation 15.16.14 gets?

Therefore, referred to the railway station, the length of the dachshund is $24/\gamma = 3.5$ inches.

15.17 *Aberration of Light*

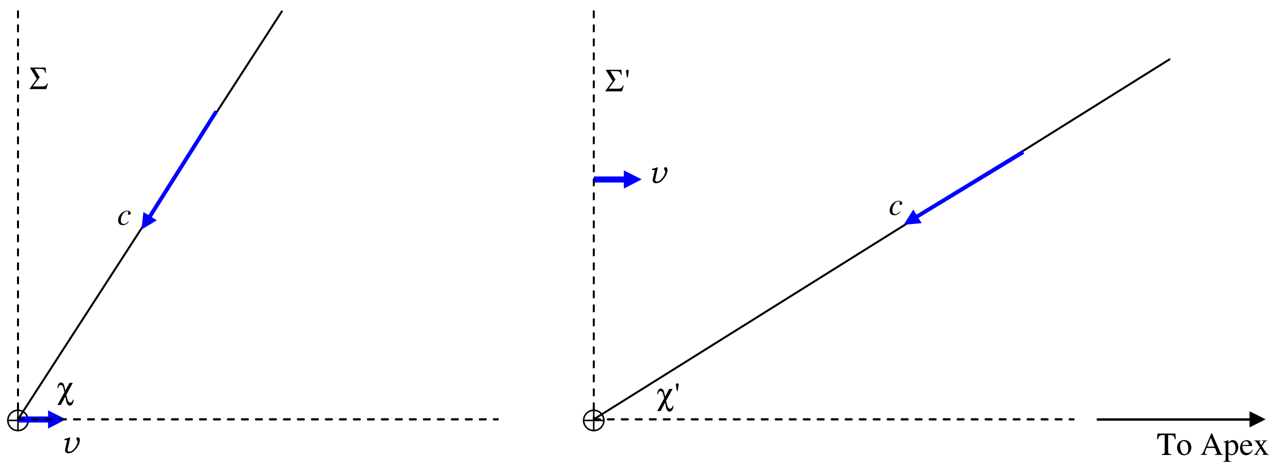


FIGURE XV.23

The direction of Earth's velocity on any particular date is called the *Apex of the Earth's Way*. In part (a) of figure XV.23 I show Earth moving towards the apex at speed υ, and light coming from a star at speed c from an angle χ from the apex. The x- and y-components of the velocity of light are respectively $-c\cos\chi$ and $-c\sin\chi$. Relative to Earth (part (b)), the x'- and y'-components are, by equations 15.16.2 and 15.16.3 (or rather their inverses)

$$-\frac{c\cos\chi + \upsilon}{1 + (\upsilon/c)\cos\chi} \text{ and } -\frac{c\sin\chi}{\gamma(1 + (\upsilon/c)\cos\chi)}.$$

You can verify that the orthogonal sum of these two components is c, as it should be according to our fundamental assumption that the speed of light is the same referred to all reference frames in uniform relative motion.

The apparent direction of the star is therefore given by

$$\sin\chi' = \frac{\sin\chi}{\gamma(1 + (\upsilon/c)\cos\chi)}. \qquad 15.17.1$$

It is left as an exercise to show that, for small υ/c, this becomes

$$\chi - \chi' = \frac{\upsilon\sin\chi}{c}. \qquad 15.17.2$$

with $\upsilon = 29.8$ km s^{-1}, υ/c is about 20".5. More details about aberration of light, including the derivation of equation 15.17.2, can be found in Celestial Mechanics, Section 11.3.

15.18 *Doppler Effect*

It is well known that the formula for the Doppler effect in sound is different according to whether it is the source or the observer that is in motion. An answer to the question "Why should this be?" to the effect that "Oh, that's just the way the algebra works out" is obviously unsatisfactory, so I shall try to explain why, physically, there is a difference. Then, when you have thoroughly understood that observer in motion is an entirely different situation from source in motion, and the formulas must be different, we shall look at the Doppler effect in light, and we'll return to square one when we find that the formulas for source in motion and observer in motion are the same!

This section on the Doppler effect will probably be rather longer than it need be, just because some aspects interested me – but if you find it too long, just skip the parts that aren't of special interest to you. These will quite likely include the parts on the *ballistic* Doppler effect.

First, we'll deal with the Doppler effect in sound. All speeds are supposed to be very small compared with the speed of light, so that we need not trouble ourselves with Lorentz transformations. First, let's deal with observer in motion (figure XV. 24).

When the source is at rest, it emits *concentric* equally-spaced spherical wavefronts at some frequency. When an observer moves towards the source, he will pass these wavefronts at a higher frequency that the frequency at which they were emitted, and that is the cause of the Doppler effect with a stationary source and moving observer.

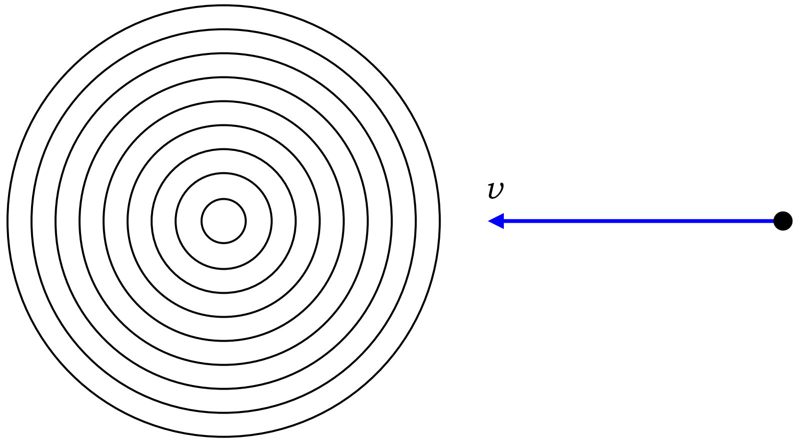


FIGURE XV.24

Now, we'll look at the source-in-motion situation. (Figure XV.25).

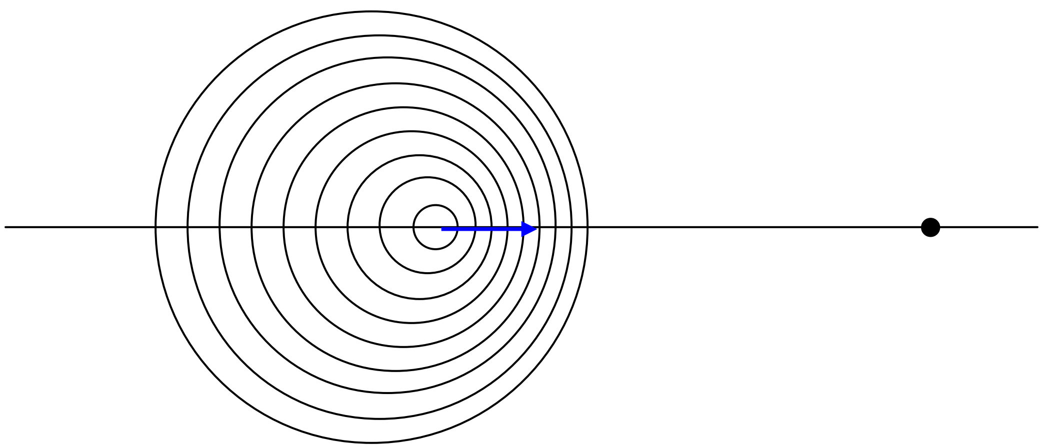


FIGURE XV.25

Here we see that the wavefronts are not equally spaced, but are compressed ahead of the motion of the source, and for that reason they will pass a stationary observer at a higher frequency than the frequency at which they were emitted. Thus the nature of the effect is a little different according to whether it is the source or the observer that is in motion, and thus one would not expect identical equations to describe the two situations.

We shall move on shortly to discuss the effect quantitatively and develop the relevant equations. I shall assume that the reader is familiar with the usual relation connecting wavelength, frequency and speed of a wave. Nevertheless I shall write down the relation in large print, three times, just to make sure:

SPEED = FREQUENCY × WAVELENGTH

FREQUENCY = SPEED ÷ WAVELENGTH

WAVELENGTH = SPEED ÷ FREQUENCY

I am going to start with the Doppler effect in <u>sound</u>, where the speed of the signal is constant with respect to the medium than transmits the sound – usually air. I shall give the necessary formulas for source *and* observer each in motion. If you want the formulas for one or the other stationary, you just put one of the speeds equal to zero. The speeds of the source S and of the observer O relative to the air will be denoted respectively by v_1 and v_2 and the speed of <u>sound</u> in air will be denoted by c. The situation is shown in figure XV.26.

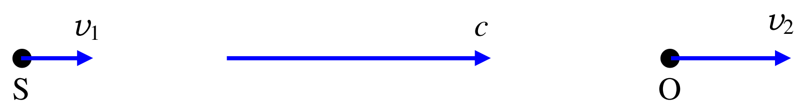


FIGURE XV.26

The relevant formulas are shown below:

	Source		Observer
Frequency	ν_0		$\nu_0\left(\dfrac{c-v_2}{c-v_1}\right)$
	↓		↑
Speed	$c - v_1$		$c - v_2$
	↓		↑
Wavelength	$(c - v_1)/\nu_0$	⟶	$(c - v_1)/\nu_0$

The way we work this table is just to follow the arrows. Starting at the top left, we suppose that the source emits a signal of frequency ν_0. The speed of the signal *relative to the source* is $c - v_1$, and so the wavelength is $(c - v_1)/\nu_0$. The wavelength is the same for the observer (we are supposing that all speeds are very much less than the speed of light, so the Lorentz factor is effectively 1.) The speed of sound relative to the observer is $c - \upsilon_2$, and so the frequency heard by the observer is the last (upper right) entry of the table.

Two special cases:
a. Observer in motion and approaching a stationary source at speed υ. $\upsilon_1 = 0$ and $\upsilon_2 = -\upsilon$. In that case the frequency heard by the observer is

$$\nu = \nu_0(1 + \upsilon/c). \tag{15.18.1}$$

b. Source in motion and approaching a stationary observer at speed υ. $\upsilon_1 = \upsilon$ and $\upsilon_2 = 0$. In that case the frequency heard by the observer is

$$\nu = \nu_0/(1 - \upsilon/c) \approx \nu_0\left(1 + (\upsilon/c) + (\upsilon/c)^2 + \ldots\right). \tag{15.18.2}$$

Thus the formulas for source in motion and observer in motion differ in the second order of (υ/c).

We might now consider *reflection*. Thus, suppose you approach a brick wall at speed υ while whistling a note of frequency ν_0. What will be the frequency of the echo that you hear? Let's make the question a little more general. A source S, emitting a whistle of frequency ν_0, approaches a brick wall M at speed υ_1. A separate observer O approaches the wall (from the same side) at speed υ_2. And, for good measure, let's have the brick wall moving at speed υ_3. (The reader may notice at this point that theoretical physics is rather easier than experimental physics.) The situation is shown in figure XV.27.

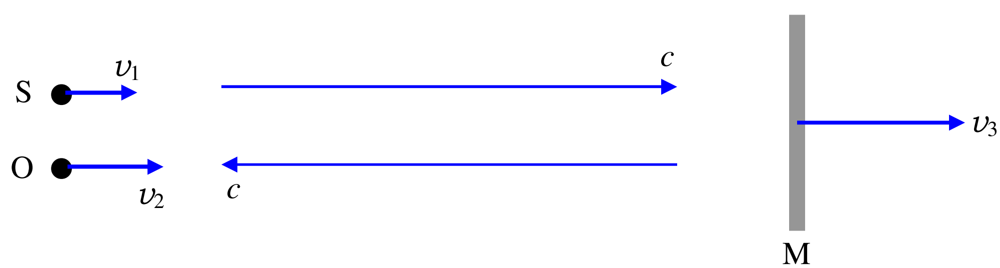


FIGURE XV.27

We construct a table similar to the previous one.

	Source	Mirror before reflection	Mirror after reflection	Observer
Frequency	ν_0	$\nu_0\left(\frac{c-v_3}{c-v_1}\right)$ →	$\nu_0\left(\frac{c-v_3}{c-v_1}\right)$	$\nu_0\left(\frac{(c+v_2)(c-v_3)}{(c-v_1)(c+v_3)}\right)$
	↓	↑	↓	↑
Speed	$c - v_1$	$c - v_3$	$c + v_3$	$c + v_2$
	↓	↑	↓	↑
Wavelength	$\frac{c-v_1}{\nu_0}$ →	$\frac{c-v_1}{\nu_0}$	$\frac{(c-v_1)(c+v_3)}{\nu_0(c-v_3)}$ →	$\frac{(c-v_1)(c+v_3)}{\nu_0(c-v_3)}$

At all times, the speed relative to the air is c.

The answer to our initial question, in which the source and the observer were one and the same, and the mirror (wall) was stationary is found by putting $v_1 = v_2 = v$ and $v_3 = 0$ in the last (top right) formula in the table. This results in

$$\nu = \nu_0\left(\frac{c+v}{c-v}\right) \approx \nu_0\left(1 + 2(v/c) + 2(v/c)^2 + 2(v/c)^3 + \ldots\right). \qquad 15.18.3$$

So much for the Doppler effect in <u>sound</u>. Before moving on to light, I want to look at what I shall call the Doppler effect in <u>ballistics</u>, or "cops and robbers". An impatient reader may safely skip this discussion of ballistic Doppler effect. A police ("cop") car is chasing a stolen car driven by robbers. The cop car is the "source" and the robber's car (or, rather the car that they have stolen, for it is not theirs) are the "observers". The cop car ("source") is travelling at speed v_1 and the robbers ("observer") is travelling at speed v_2. The cops are firing bullets (the "signal") towards the robbers. (No one gets hurt in this thought experiment, which is all make-believe.) The bullets leave the muzzle of the revolver at speed c (that is the speed of the *bullets*, and is nothing to do with *light*) relative to the revolver, and hence they travel (relative to the lamp-posts at the side of the road) at speed $c + v_1$ and relative to the robbers at speed $c + v_1 - v_2$. The cops fire bullets at frequency ν_0, and our task is to find the frequency with which the bullets are "received" by the robbers. The distance between the bullets is the "wavelength".

This may not be a very important exercise, but it is not entirely pointless, for fairness dictates that, when we are considering (even if only to discard) possible plausible mechanisms for the propagation of light, we might consider, at least briefly, the so-called "ballistic" theory of light propagation, in which the speed of light through space is equal to the speed at which it leaves the source plus the speed of the source. Some readers may be aware of the Michelson-Morley experiment. That experiment demonstrated that light was not propagated at a speed that was constant with respect to some all-pervading "luminiferous aether" – but it must be noted that it did nothing to prove or disprove the "ballistic" theory of light propagation, since it did not measure the speed of light from moving sources. In the intervening years, some attempts have indeed been made to measure the speed of light from moving sources, though their interpretation has not been free from ambiguity.

I now construct a table showing the "frequency", "speed" and "wavelength" for ballistic propagation in exactly the same way as I did for sound.

	Source	Observer
Frequency	ν_0	$\nu_0\left(\dfrac{c+v_1-v_2}{c}\right)$
Speed	c	$c + v_1 - v_2$
Wavelength	c/ν_0	c/ν_0

In order not to spend longer on "ballistic" propagation than is warranted by its importance, I'll just let the reader spend as much or as little time pondering over this table as he or she wishes. Just one small point might be noted, namely that the formulas for "observer in motion" and "source in motion" are the same.

For completeness rather than for any important application, I shall also construct here the table for "reflection". A source of bullets is approaching a mirror at speed v_1. An observer is also approaching the mirror, from the same side, at speed v_2. And the mirror is moving at speed v_3, and reflection is elastic (the coefficient of restitution is 1.) You are free to put as many of these speeds equal to zero as you wish.

The entries for "speed" give the speed relative to the source or mirror or observer. The speed relative to stationary lampposts at the side of the road is $c + v_1$ before reflection and $c + v_1 - 2v_3$ after reflection.

	Source	Mirror before reflection	Mirror after reflection	Observer
Frequency	ν_0	$\nu_0\left(\frac{c+v_1-v_3}{c}\right)$ →	$\nu_0\left(\frac{c+v_1-v_3}{c}\right)$	$\nu_0\left(\frac{c+v_1+v_2-2v_3}{c}\right)$
Speed	c	$c + v_1 - v_3$	$c + v_1 - v_3$	$c + v_1 + v_2 - 2v_3$
Wavelength	$\frac{c}{\nu_0}$ →	$\frac{c}{\nu_0}$	$\frac{c}{\nu_0}$ →	$\frac{c}{\nu_0}$

We now move on to the only aspect of the Doppler effect that is really relevant to this chapter, namely the Doppler effect in light. In the previous two situations I have been able to assume that all speeds were negligible compared with the speed of light, and we have not had to concern ourselves with relativistic effects. Here, however, the signal *is* light and is propagated at the speed of light, and this speed is the same whether referred to the reference frame in which the source is stationary or the observer is stationary. Further, the Doppler effect is noticeable only if source or observer are moving at speeds comparable to that of light. We shall see that the difference between the frequency of a signal relative to an observer and the frequency relative to the source is the result of *two* effects, which, while they may be treated separately, are both operative and in that sense inseparable. These two effects are the *Doppler effect proper*, which is a result of the changing distance between source and observer, and the relativistic *dilation of time*.

I am going to use the symbol T to denote the time interval between passage of consecutive crests of an electromagnetic wave. I'll call this the *period*. This is merely the reciprocal of the frequency ν. I am going to start by considering a situation in which a source and an observer a receding from each other at a speed v. I have drawn this in figure XV.27, which is referred to a frame in which the observer is at rest. The speed of light is c.

FIGURE XV.27

Let us suppose that S emits an electromagnetic wave of period $T_0 = 1/\nu_0$ *referred to the frame in which S is at rest.* We are going to have to think about *four* distinct periods or frequencies:

1. The time interval between the emission of consecutive crests by S *referred to the reference frame in which S is at rest.* This is the period T_0 and the frequency ν_0 that we have just mentioned.

2. The time interval between the emission of consecutive crests by S *referred to the reference frame in which O is at rest.* By the relativistic formula for the dilation of time this is

$$\gamma T_0 \quad or \quad \frac{T_0}{\sqrt{1-\upsilon^2/c^2}}. \qquad 15.18.4$$

3. The time interval between the reception of consecutive crests by O as a result of the increasing distance between O and S (the "true" Doppler effect, as distinct from time dilation) *referred to the reference frame in which S is at rest.* This is

$$T_0(1+\upsilon/c). \qquad 15.18.5$$

4 The time interval between the reception of consecutive crests by O as a result of the increasing distance between O and S (the "true" Doppler effect, as distinct from time dilation) *referred to the reference frame in which O is at rest.* This is

$$\gamma \text{ times } T_0(1+\upsilon/c). \qquad 15.18.6$$

This, of course, is what O "observes", and, when you do the trivial algebra, you find that this is

$$T = T_0\sqrt{\frac{1+\upsilon/c}{1-\upsilon/c}}, \qquad 15.18.7$$

or, in terms of frequency,

$$\nu = \nu_0\sqrt{\frac{1-\upsilon/c}{1+\upsilon/c}}. \qquad 15.18.8$$

If source and observer *approach* each other at speed υ, the result is

$$\nu = \nu_0\sqrt{\frac{1+\upsilon/c}{1-\upsilon/c}}. \qquad 15.18.9$$

The factor $\sqrt{\frac{1+\upsilon/c}{1-\upsilon/c}}$ is often denoted by the symbol k, and indeed that was the symbol I used in section 15.3 (see equation 15.3.3).

Exercise. Expand equation 15.18.9 by the binomial theorem as far as $(\upsilon/c)^2$ and compare the result with equations 15.8.1 and 15.8.2.

I make it $$\nu = \nu_0\left(1 + (\upsilon/c) + \tfrac{1}{2}(\upsilon/c)^2 \ldots\right). \qquad 15.18.10$$

Question.

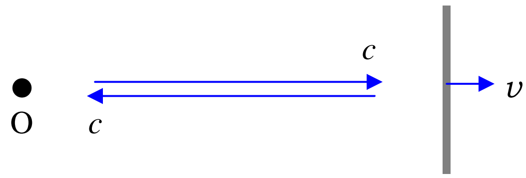


FIGURE XV.28

An observer O sends an electromagnetic signal of frequency ν_0 at speed c to a mirror that is receding at speed υ. When the reflected signal arrives back at the observer, what is its frequency (to first order in υ/c)? Is it $\nu_o(1-\upsilon/c)$ or is it $\nu_o(1-2\upsilon/c)$? I can think offhand of two applications of this. If you examine the solar Fraunhofer spectrum reflected of the equatorial limb of a rotating planet, and you observe the fractional change $\Delta\nu/\nu_0$ in the frequency of a spectrum line, will this tell you υ/c or $2\upsilon/c$, where υ is the equatorial speed of the planet's surface? And if a policeman directs a radar beam at your car, does the frequency of the returning beam tell him the speed of your car, or twice its speed? You could try arguing this case in court – or, better, stick to the speed limit so there is no need to do so. The answer, by the way, is $\nu_o(1-2\upsilon/c)$.

Redshift. When a galaxy is moving away from us, a spectrum line of laboratory wavelength λ_0 will appear to have a frequency for the observer of $\lambda = k\lambda_0$. The fractional increase in wavelength $\dfrac{\lambda-\lambda_0}{\lambda_0}$ is generally given the symbol z, which is evidently equal to $k - 1$. (Only to first order in β is it approximately equal to β. It is important to note that the definition of z is $\dfrac{\lambda-\lambda_0}{\lambda_0}$, and not υ/c.

A note on terminology: If a source is receding from the observer the light is observed to be shifted towards longer wavelengths, and if it is approaching the observer the light is shifted towards shorter wavelengths. Traditionally a shift to longer wavelengths is called a "redshift", and a shift towards shorter wavelengths is called a "blueshift". Note, however, that if an infrared source is approaching an observer, its light is shifted towards the red, and if an ultraviolet source is receding from an observer, its light is shifted towards the blue! Nevertheless I shall continue in this chapter to refer to shifts to longer and shorter wavelengths as redshifts and blueshifts respectively.

Example.

FIGURE XV.29

A red galaxy R of wavelength 680.0 nm and a green galaxy G of wavelength 520.0 nm are on opposite sides of an observer X, both receding from him/her. To the observer, the wavelength of the red galaxy appears to be 820.0 nm, and the wavelength of the green galaxy appears to be 640.0 nm. What is the wavelength of the green galaxy as seen from the red galaxy?

Solution. We are told that k for the red galaxy is 82/68 = 1.20588, or z = 0.20588, and that k for the green galaxy k is 64/52 = 1.23077, or z = 0.23077. Because of the preparation we did in section 15.3, we can instantly convert these to ϕ. Thus for the red galaxy $\phi = 0.187212$ and for the green galaxy $\phi = 0.207639$. The sum of these is 0.394851. We can instantly convert this to $k = 1.48416$ or $z = 0.48416$. Thus, as seen from R, the wavelength of G is 771.8 nm.

Alternatively. Show that the factor k combines as

$$k_1 \oplus k_2 = k_1 k_2 \qquad 15.18.11$$

and verify that $\frac{82}{68} \times \frac{64}{52} = 1.48416$. Show also that the redshift factor z combines as

$$z_1 \oplus z_2 = z_1 z_2 + z_1 + z_2 . \qquad 15.18.12$$

15.19 *The Transverse and Oblique Doppler Effects*

I pointed out in section 15.18 that the observed Doppler effect, when the transmitted signal is electromagnetic radiation and observer or source or both are travelling at speed comparable to that of light, is a combination of two effects – the "true" Doppler effect, caused by the changing distance between source and observer, and the effect of time dilation. This raises the following question:

If a source of light is moving at right angles to (transverse to) the line joining observer to source, will the observer see a change in frequency or wavelength, even though the distance between observer and source at that instant is not changing? The answer is yes, certainly, and the effect is sometimes called the "transverse Doppler effect", although it is the effect of relativistic time dilation rather than of a true Doppler effect.

Thus let us suppose that a source is moving transverse to the line of sight at a speed described by its parameter β or γ, and that the period of the radiation referred to the reference frame in which the source is at rest is T_0 and the frequency is ν_0. The time interval between emission of consecutive wavecrests when referred to the frame in which the observer is at rest is longer by the gamma-factor, and the frequency is correspondingly less. That is, the frequency, referred to the observer's reference frame, is

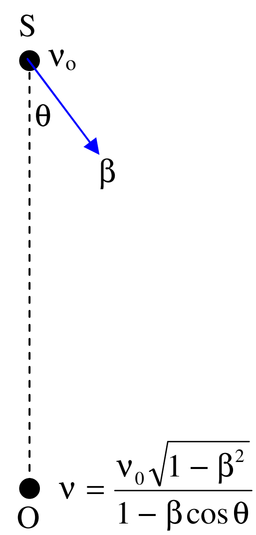
$$\nu = \nu_0 / \gamma = \nu_0 \sqrt{1 - \beta^2} . \qquad 15.19.1$$

The light from the source is therefore seen by the observer to be redshifted, even though there is no radial velocity component.

This raises a further question. Suppose a source is moving almost but not quite at right angles to the line of sight, so that it has a large transverse velocity component, and a small velocity component towards the observer. In that case, its "redshift" resulting from the time dilation might be appreciable, while its "blueshift" resulting from "true" Doppler effect (the decreasing distance between source and observer) is still very small. Therefore, even though the distance between source and observer is slightly decreasing, there is a net redshift of the spectrum. This is in fact correct, and is the "oblique Doppler effect".

In figure XV.30, a source S is moving at speed β times the speed of light in a direction that makes an angle θ with the line of sight. It is emitting a signal of frequency ν_0 in S. (I am here using the frame "in S" as earlier in the chapter to mean "referred to a reference frame in which S is at rest.) The signal arrives at the observer O at a slightly greater frequency as a result of the decreasing distance of S from O, and at a slightly lesser frequency as a result of the time dilation, the two effects opposing each other.

FIGURE XV.30



The frequency of the received signal at O, in O, is

$$\nu = \frac{\nu_0\sqrt{1-\beta^2}}{1-\beta\cos\theta}. \qquad 15.19.2$$

For a given angle θ the redshift is zero for a speed of

$$\beta = \frac{2\cos\theta}{1+\cos^2\theta} \qquad 15.19.3$$

or, for a given speed, the direction of motion resulting in a zero redshift is given by

$$\beta \cos^2 \theta - 2 \cos \theta + \beta = 0. \qquad 15.19.4$$

This relation is shown in figure XV.30. (Although equation 15.19.4 is quadratic in cos θ, there is only one real solution θ for β between 0 and 1. Prove this assertion.) It might be noted that if the speed of the source is 99.99% of the speed of light the observer will see a redshift unless the direction of motion of S is no further than 9° 36' from the line from S to O. That is worth repeating: S is moving very close to the speed of light in a direction that is close to being directly towards the observer; the observer will see a redshift.

Equation 15.19.2, which gives ν as a function of θ for a given β, will readily be recognized at the equation of an ellipse of eccentricity β, semi minor axis ν_0 and semi major axis $\gamma\nu_0$. This relation is shown in figure XV.31 for several β. The curves are red where there is a redshift and blue where there is a blueshift. There is no redshift or blueshift for β = 0, and the ellipse for that case is a circle and is drawn in black.

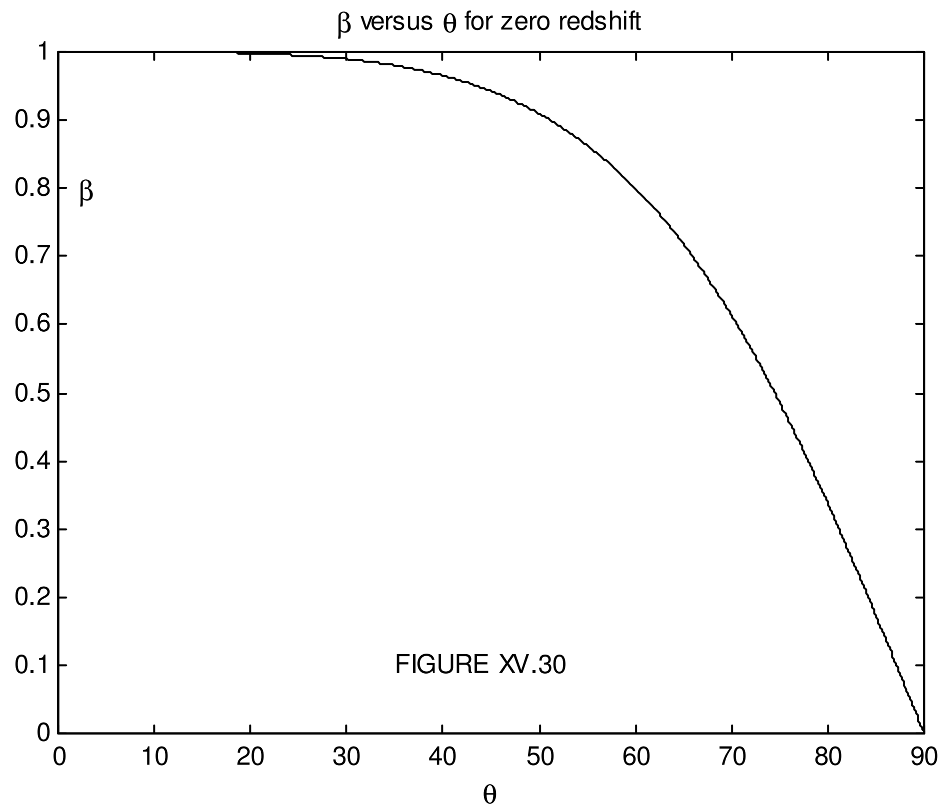


FIGURE XV.30

An alternative and perhaps more useful way of looking at equation 15.19.2 is to regard it as an equation that gives β as a function of θ for a given Doppler ratio ν/ν_0. For example, if the Doppler ratio of a galaxy is observed to be 0.75, the velocity vector of the galaxy could be any arrow starting at the black dot and ending on the curve marked 0.75. The curves are ellipses with semi major axis equal to $1/\sqrt{1-(\nu/\nu_0)^2}$ and semi minor axis $1/\left(1-(\nu/\nu_0)^2\right)$.

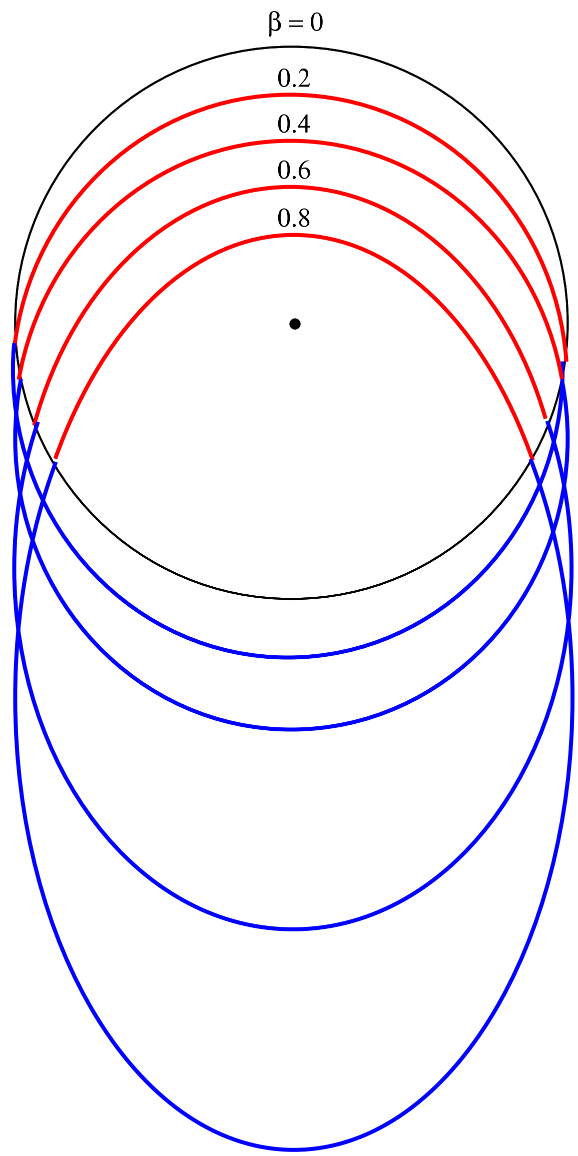


FIGURE XV.31

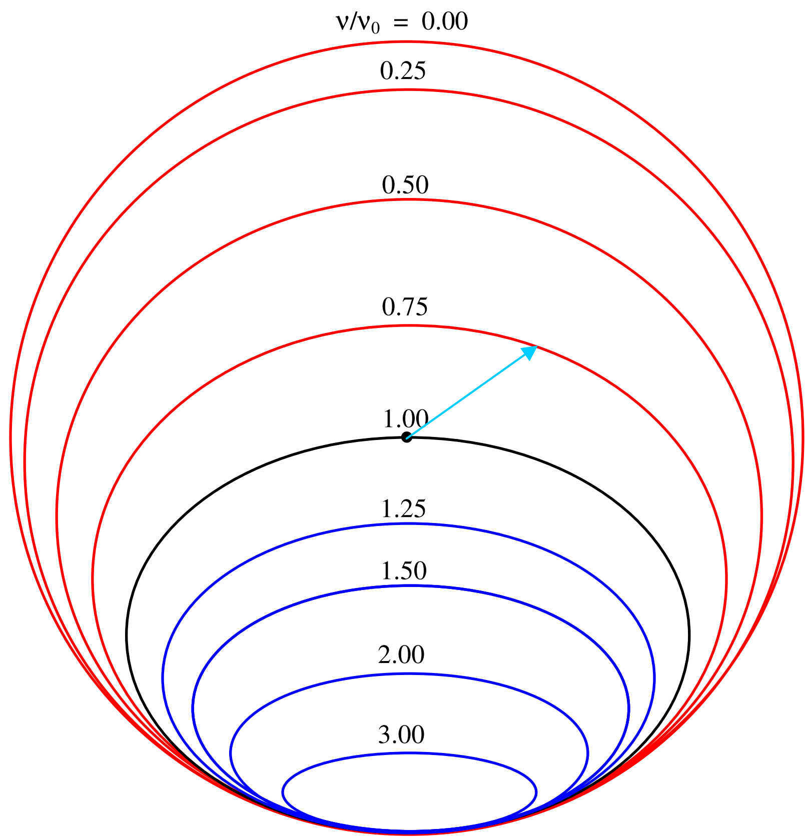


FIGURE XV.32

15.20 *Acceleration*

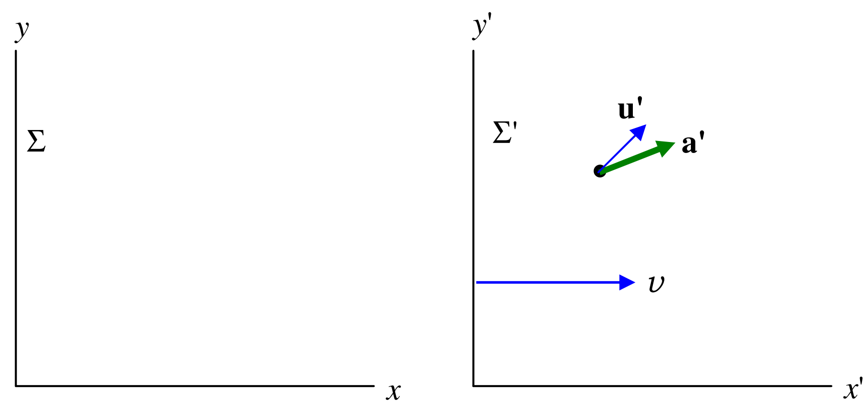


FIGURE XV.33

Figure XV.33 shows two references frames, Σ and Σ', the latter moving at speed υ with respect to the former. A particle is moving with acceleration **a'** in Σ'. (" in Σ' " = "referred to the reference frame Σ' ".) The velocity is not necessarily, of course, in the same direction as the acceleration, and we'll suppose that its velocity in Σ' is **u'**. The acceleration and velocity components in Σ' are $a'_{x'}, a'_{y'}, u'_{x'}, u'_{y'}$.

What is the acceleration of the particle in Σ? We shall start with the *x*- component.

The *x*-component of its acceleration in Σ is given by

$$a_x = \frac{du_x}{dt}, \qquad 15.20.1$$

where $$u_x = \frac{u'_x + \upsilon}{1 + u'_x \upsilon / c^2} \qquad 15.16.2$$

and $$t = \gamma(t' + \upsilon x'/c^2) . \qquad 15.5.19$$

Equations 15.16.2 and 15.5.19 give us

$$du_x = \frac{du_x}{du'_x} du'_x = \frac{du'_x}{\gamma^2 (1 + u'_x \upsilon / c^2)^2} \qquad 15.20.2$$

and $$dt = \frac{\partial t}{\partial t'} dt' + \frac{\partial t}{\partial x'} dx' = \gamma dt' + \frac{\gamma \upsilon}{c^2} dx'. \qquad 15.20.3$$

On substitution of these into equation 15.20.1 and a very little algebra, we obtain

$$a_x = \frac{a'}{\gamma^3 (1 + u'_x \upsilon / c^2)^3} . \qquad 15.20.4$$

The *y*-component of its acceleration in Σ is given by

$$a_y = \frac{du_y}{dt}, \qquad 15.20.5$$

We have already worked out the denominator *dt* (equation 15.20.3). We know that

$$u_y = \frac{u'_{y'}}{\gamma (1 + u'_{x'} \upsilon / c^2)} , \qquad 15.16.3$$

from which

$$du_y = \frac{\partial u_y}{\partial u'_{x'}}du'_{x'} + \frac{\partial u_y}{\partial u'_{y'}}du'_{y'} = \frac{1}{\gamma}\left(-\frac{\upsilon u'_{y'}}{c^2(1+\upsilon u'_{x'}/c^2)^2}du'_{x'} + \frac{1}{1+\upsilon u'_{x'}/c^2}du'_{y'}\cdot\right)$$

15.20.6

Divide equation 15.20.6 by equation 15.20.3 to obtain

$$a_y = \frac{1}{\gamma^2}\left(-\frac{\upsilon u'_{x'}}{c^2(1+\upsilon u'_{x'}/c^2)^3}a'_{x'} + \frac{1}{(1+\upsilon u'_{x'}/c^2)^2}a'_{x'}\right). \qquad 15.20.7$$

21. *Mass*

It is well known that "in relativity" the mass of an object increases as its speed increases. This may be well known, but I am not certain that it is a very precise statement of the true situation. Or at least it is no more precise than to say that the length of a rod decreases as its speed increases. The length of a rod when referred to a frame in which it is at rest is called its *proper length* l_0, and the mass of a body when referred to a frame in which it is at rest is called its *rest mass* m_0, and both of these things are invariant. The length of a rod when referred to a reference frame that is moving with respect to it (i.e., in Minkowski language, its component along an inclined axis) and the mass of a body referred to a frame that is moving with respect to it may indeed be different from the proper length of the rod or the rest mass of the body.

In order to derive the FitzGerald-Lorentz contraction, we had to think about what we mean by "length" and how to measure it. Likewise, in order to derive the "relativistic increase of mass" (which may be a misnomer) we have to think about what we mean by mass and how to measure it.

The fundamental unit of mass used at present in science is the International Prototype Kilogram, a platinum-iridium alloy, held in Sèvres, Paris, France. In order to determine the mass, or inertia, of another body, we need to carry out an experiment to compare its reluctance to accelerate when a force is applied to it with the reluctance of the standard kilogram when the same force is applied. We might, for example, attach the body to a spring, stretch the spring, let go, and see how fast the body accelerates. Then we carry out the same experiment with the International Prototype Kilogram. Or we might apply an impulse ($\int I dt$ - see Chapter 8) to the body and to the Kilogram, and measure the speed immediately after applying the impulse. This might be done, for example, but striking the body and the Kilogram with a golf club, or, for a more controlled experiment, one could press each body up against a compressed spring, release the spring, and measure the resulting speed imparted to the body and to the Kilogram. (It is probable that the International Prototype Kilogram is kept under some sort of guard, and its curators may not altogether appreciate such experiments, so perhaps these experiments had better remain Thought Experiments.) Yet another method would be to cause the body and the Kilogram to collide with each other, and to assume that the collision is elastic (no internal degrees of freedom) and that momentum (defined as the product of mass and velocity) are conserved.

All of these experiments measure the reluctance to accelerate under a force; in other words the *inertia* or the *inertial mass* or just the *mass* of the body.

Another possible experiment to determine the mass of the body would be to place it and the Kilogram at a measured distance from another mass (such as the Earth) and measure the gravitational force (weight) of each. One has an uneasy feeling that this sort of measurement is somehow a little different from the others, in that it isn't a measure of *inertia*. Some indeed would differentiate between the *inertial mass* and the *gravitational mass* of a body, although the two are in fact observed to be strictly proportional to one another. Some would not find the proportionality between inertial and gravitational mass particularly remarkable; to others, the proportionality is a surprising fact of the profoundest significance.

In this chapter we do not deal with general relativity or with gravity, so we shall think of mass in terms of its inertia. I am going to measure the ratio of two masses (one of which might be the International Prototype Kilogram) by allowing them to collide, and their masses are to be defined by assuming that the momentum of the system is conserved in all uniformly moving reference frames.

Figure XV.34 shows two references frames, Σ and Σ', the latter moving to the right at speed v relative to the former. Two bodies, of identical masses in Σ' (i.e. referred to the frame Σ'), are moving at speed u' in Σ', one of them to the right, the other to the left. Their mutual centre of mass is stationary in Σ'.

FIGURE XV.34

Now let us refer the situation to the frame Σ (see figure XV.35).

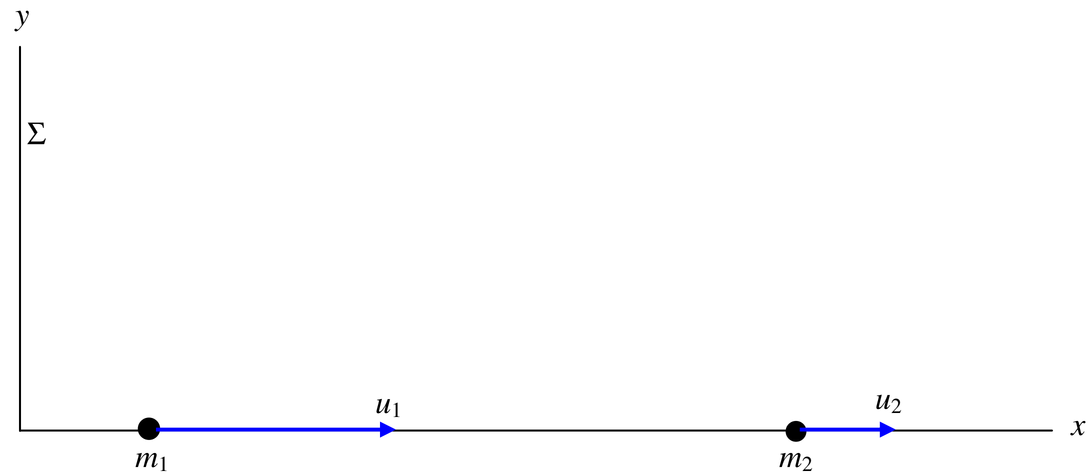


FIGURE XV.35

The total momentum of the system in Σ is $m_1u_1 + m_2u_2$. But the centre of mass (which is stationary in Σ') is moving to the right in Σ with speed υ. Therefore the momentum is also $(m_1 + m_2)\upsilon$. If they stick together upon collision, we are left with a single particle of mass $m_1 + m_2$ moving at speed υ, and, because there are no external forces, the momentum is conserved. In any case, whether the collision is elastic or not, we have

$$m_1u_1 + m_2u_2 = (m_1 + m_2)\upsilon . \qquad 15.21.1$$

But
$$u_1 = \frac{u' + \upsilon}{1 + u'\upsilon / c^2} \quad \text{and} \quad u_2 = \frac{-u' + \upsilon}{1 - u'\upsilon / c^2} . \qquad 15.21.2\text{a,b}$$

Our aim is to try to find a relation between the masses and speeds referred to Σ. Therefore we must eliminate υ and u' from equations 15.21.1, 15.21.2a and 15.21.2b. This can be a bit fiddly, but the algebra is straightforward, and I leave it to the reader to show that the result is

$$\frac{m_1}{m_2} = \sqrt{\frac{1 - u_2^2 / c^2}{1 - u_1^2 / c^2}} . \qquad 15.21.3$$

This tells us that the mass m of a body referred to Σ is proportional to $1/\sqrt{1 - u^2 / c^2}$, where u is its speed referred to Σ. If we call the proportionality constant m_0, then

$$m = \frac{m_0}{\sqrt{1 - u^2/c^2}}\,. \qquad 15.21.4$$

If $u = 0$, then $m = m_0$, and m_0 is called the *rest mass*, and it is the mass when referred to a frame in which the body is at rest. The mass m is generally called the *relativistic mass*, and it is the mass when referred to a frame in which the speed of the body is u.

Equation 15.21.4 gives the mass referred to Σ assuming that the mass is at rest in Σ'. But what if the mass is not at rest in Σ'?

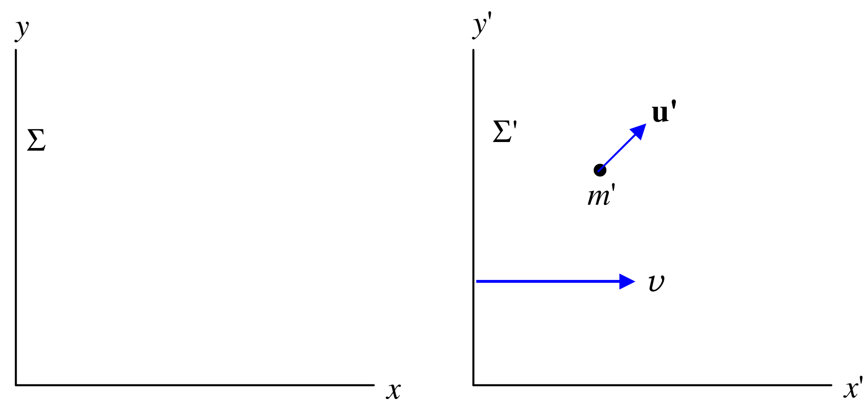


FIGURE XV.36

In figure XV.36 we see a mass m' moving with velocity **u**' in Σ'. Referred to Σ its mass will be m, where

$$\frac{m}{m'} = \sqrt{\frac{1 - u'^2/c^2}{1 - u^2/c^2}}\,. \qquad 15.21.5$$

Its velocity **u** will be in a different direction (referred to Σ) from the direction of **u'** in Σ', and the speed will be given by

$$u^2 = u_x^2 + u_y^2, \qquad 15.21.6$$

where u_x and u_y are given by equations 15.16.2 and 15.16.3. Substitute equations 15.21.6, 15.16.2 and15.16.3 into equation 15.21.5. The objective is to replace u entirely by primed quantities. The algebra is slightly boring, but it is worth persisting. You will find that $u'^2_{y'}$ appears when you use equation 15.16.3. Replace that by $u'^2 - u'^2_{x'}$. Also write $1/(1 - \upsilon^2/c^2)$ for γ^2. After a little while you should arrive at

$$\frac{m}{m'} = \gamma\left(1 + \frac{vu'_{x'}}{c^2}\right). \qquad 15.21.7$$

The transformation for mass between the two frames depends only on the x' component of its velocity in Σ'. It would have made no difference, other than to increase the tedium of the algebra, if I had added $+u_z^2$ to the right hand side of equation 15.21.6.

The inverse of equation 15.21.7 is found in the usual way by interchanging the primed and unprimed quantities and changing the sign of v:

$$\frac{m'}{m} = \gamma\left(1 - \frac{v\,u_x}{c^2}\right). \qquad 15.21.8$$

Example. Let's return to the problem of the dachshund that we met in section 15.16. A railway train Σ' is trundling along at a speed v/c = 0.9 (γ = 2.294). The dachshund is waddling towards the front of the train at a speed $u'_{x'}/c = 0.8$. In the reference frame of the train Σ' the mass of the dog is m' = 8 kg. In the reference frame of the railway station, the mass of the dog is given by equation 15.21.7 and is 31.6 kg. (Its length is also much compressed, so it is very dense when referred to Σ and is disc-shaped.) I leave it to the reader to show that the rest mass of the dog is 4.8 kg.

15.22 *Momentum*

The linear momentum **p** of a body, referred to a frame Σ, is defined as

$$\mathbf{p} = m\mathbf{u}. \qquad 15.22.1$$

Here m and **u** are its mass and velocity referred to Σ. Note that m is not the rest mass.

Example. The rest mass of a proton is 1.67×10^{-27} kg. What is its momentum referred to a frame in which it is moving at 99% of the speed of light? Answer = 3.51×10^{-18} kg m s^{-1}.

15.23 *Some Mathematical Results*

Before proceeding with the next section, I just want to establish few mathematical results, so that we don't get bogged down in heavy algebra later on when we should be concentrating on understanding physics.

First, if
$$\gamma = (1 - u^2/c^2)^{-1/2}, \qquad 15.23.1$$

Then, by trivial differentiation,

$$\frac{d\gamma}{du} = \frac{\gamma^3 u}{c^2}. \qquad 15.23.2$$

$$\therefore \qquad \dot{\gamma} = \frac{\gamma^3 u \dot{u}}{c^2}. \qquad 15.23.3$$

From this, we quickly find that

$$\frac{\gamma u \dot{u}}{\dot{\gamma}} = c^2 - u^2. \qquad 15.23.4$$

Now for a small result concerning a scalar (dot) product.

Let **A** be a vector such that $\mathbf{A} \bullet \mathbf{A} = A^2$.

Then $\quad \frac{d}{dt}(A^2) = 2A\dot{A} \quad$ *and* $\quad \frac{d}{dt}(\mathbf{A} \bullet \mathbf{A}) = 2\mathbf{A} \bullet \dot{\mathbf{A}}$

$$\therefore \qquad \mathbf{A} \bullet \dot{\mathbf{A}} = A\dot{A}. \qquad 15.23.5$$

We can now safely proceed to the next section.

15.24 *Kinetic Energy*

If a force **F** acts on a particle moving with velocity **u**, the rate of doing work – i.e. the rate of increase of kinetic energy T is $\dot{T} = \mathbf{F} \bullet \mathbf{u}$. But $\mathbf{F} = \dot{\mathbf{p}}$, where $\mathbf{p} = m\mathbf{u} = \gamma m_0 \mathbf{u}$. (A point about notation may be in order here. I have been using the symbol $\boldsymbol{\upsilon}$ and υ for the velocity and speed of a frame Σ' relative to a frame Σ, and my choice of axes without significant loss of generality has been such that $\boldsymbol{\upsilon}$ has been directed parallel to the x-axis. I have been using the symbol **u** for the velocity (speed = u) of a particle relative to the frame Σ. Usually the symbol γ has meant $(1 - \upsilon^2/c^2)^{-1/2}$, but here I am using it to mean $(1 - u^2/c^2)^{-1/2}$. I hope that this does not cause too much confusion and that the context will make it clear. I toyed with the idea of using a different symbol, but I thought that this might make matters worse. Just be on your guard, anyway.)

We have, then

$$\mathbf{F} = m_0(\dot{\gamma}\mathbf{u} + \gamma\dot{\mathbf{u}}) \qquad 15.24.1$$

and therefore

$$\dot{T} = m_0(\dot{\gamma}u^2 + \gamma\dot{\mathbf{u}} \bullet \mathbf{u}). \qquad 15.24.2$$

Making use of equations 15.23.4 and 15.23.5 we obtain

$$\dot{T} = \dot{\gamma} m_0 c^2. \qquad 15.24.3$$

Integrate with respect to time, with the condition that when $\gamma = 1$, $T = 0$, and we obtain the following expression for the kinetic energy:

$$T = (\gamma - 1)m_0 c^2. \qquad 15.24.4$$

Exercise. Expand γ by the binomial theorem as far as u^2/c^2, and show that, to this order, $T = \frac{1}{2}m_0 u^2$.

I here introduce the dimensionless symbol

$$K = \frac{T}{m_0 c^2} = \gamma - 1 \qquad 15.24.5$$

to mean the kinetic energy in units of m_0c^2. The second half of this was already given as equation 15.3.5.

15.25 *Addition of Kinetic Energies*

I want now to consider two particles moving at nonrelativistic speeds – by which I mean that the kinetic energy is given to a sufficient approximation by the expression $\frac{1}{2}mu^2$ and so that parallel velocities add linearly.

Consider the particles in figure XV.37, in which the velocities are shown relative to laboratory space.

FIGURE XV.37

Referred to laboratory space, the kinetic energy is $\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2$. However, the centre of mass is moving to the right with speed $V = (m_1u_1 + m_2u_2)/(m_1 + m_2)$, and, referred to centre of mass space, the kinetic energy is $\frac{1}{2}m_1(u_1 - V)^2 + \frac{1}{2}m(u_2 + V)^2$. On the other hand, if we refer the situation to a frame in which m_1 is at rest, the kinetic energy is $\frac{1}{2}m_2(u_1 + u_2)^2$, and, if we refer the situation to a frame in which m_2 is at rest, the kinetic energy is $\frac{1}{2}m_1(u_1 + u_2)^2$.

All we are saying is that the kinetic energy depends on the frame to which speed are referred – and this is not something that crops up only for relativistic speeds.

Let us put some numbers in. Let us suppose, for example that

$$m_1 = 3 \text{ kg} \qquad u_1 = 4 \text{ m s}^{-1}$$

$$m_2 = 2 \text{ kg} \qquad u_3 = 4 \text{ m s}^{-1}$$

so that $$V = 1.2 \text{ m s}^{-1}.$$

In that case, the kinetic energy

referred to laboratory space is	33 J,
referred to centre of mass space is	29.4 J,
referred to m_1 is	49 J,
referred to m_2 is	73.5 J.

In this case the kinetic energy is least when referred to centre of mass space, and is greatest when referred to the lesser mass.

Exercise. Is this always so, whatever the values of m_1, m_2 , u_1 and u_2?

It may be worthwhile to look at the special case in which the two masses are equal (m) and the two speeds(whether in laboratory or centre of mass space) are equal (u).

In that case the kinetic energy in laboratory or centre of mass space is mu^2, while referred to either of the masses it is $2mu^2$.

We shall now look at the same problem for particles travelling at relativistic speeds, and we shall see that the kinetic energy referred to a frame in which one of the particles is at rest is very much greater than (not merely twice) the energy referred to a centre of mass frame.

If two particles are moving towards each other with "speeds" given by γ_1 and γ_2 in centre of mass space, the γ of one relative to the other is given by equation 15.16.14, and, since $K = \gamma - 1$, it follows that if the two particles have kinetic energies K_1 and K_2 in centre of mass space (in units of the $m_0\text{c}^2$ of each), then the kinetic energy of one relative to the other is

$$K = K_1 \oplus K_2 = K_1 + K_2 + K_1 K_2 + \sqrt{K_1 K_2 (K_1 + 2)(K_2 + 2)}. \qquad 15.25.1$$

If two identical particles, each of kinetic energy K_1 times m_0c^2, approach each other, the kinetic energy of one relative to the other is

$$K = 2K_1(K_1 + 2). \qquad 15.25.2$$

For nonrelativistic speeds as $K_1 \rightarrow 0$, this tends to $K = 4K_1$, as expected.

Let us suppose that two protons are approaching each other at 99% of the speed of light in centre of mass space (K_1 = 6.08881). Referred to a frame in which one proton is at rest, the kinetic energy of the other will be K = 98.5025, the relative speeds being 0.99995 times the speed of light. Thus $K = 16K_1$ rather than merely $4K_1$ as in the nonrelativistic calculation. For more energetic particles, the ratio K/K_1 is even more. These calculations are greatly facilitated if you wrote, as suggested in section 15.3, a program that instantly connects all the relativity factors given there.

Exercise. Two protons approach each other, each having a kinetic energy of 500 GeV in laboratory or centre of mass space. (Since the two rest masses are equal, these two spaces are identical.) What is the kinetic energy of one proton in a frame in which the other is at rest? (Answer: I make it 535 TeV.)

The factor K (the kinetic energy in units of m_0c^2) is the last of several factors used in this chapter to describe the speed at which a particle is moving, and I take the opportunity here of summarising the formulas that have been derived in the chapter for combining these several measures of speed. These are

$$\beta_1 \oplus \beta_2 = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}. \qquad 15.16.7$$

$$\gamma_1 \oplus \gamma_2 = \gamma_1\gamma_2 + \sqrt{(\gamma_1^2 - 1)(\gamma_2^2 - 1)}. \qquad 15.16.14$$

$$k_1 \oplus k_2 = k_1k_2 \qquad 15.18.11$$

$$z_1 \oplus z_2 = z_1z_2 + z_1 + z_2. \qquad 15.18.12$$

$$K = K_1 \oplus K_2 = K_1 + K_2 + K_1K_2 + \sqrt{K_1K_2(K_1 + 2)(K_2 + 2)}. \qquad 15.25.1$$

$$\phi_2 \oplus \phi_2 = \phi_1 + \phi_2. \qquad 15.16.11$$

If the two speeds to be combined are equal, these become

$$\beta_1 \oplus \beta_1 = \frac{2\beta_1}{1 + \beta_1^2}. \qquad 15.25.3$$

$$\gamma_1 \oplus \gamma_1 = 2\gamma_1^2 - 1 \qquad 15.25.4$$

$$k_1 \oplus k_1 = k_1^2 \qquad 15.25.5$$

$$z_1 \oplus z_1 = z_1(z_1 + 2) \qquad 15.25.6$$

$$K_1 \oplus K_1 = 2K_1(K_1 + 2). \qquad 15.25.7$$

$$\phi_1 \oplus \phi_1 = 2\phi. \qquad 15.25.8$$

These formulas are useful, but for numerical examples, if you already have a program for interconverting between all of these factors, the easiest and quickest way of combinng two "speeds" is to convert them to ϕ. We have seen examples of how this works in sections 15.16 and 15.18. We can do the same thing with our example from the present section when combining two kinetic energies. Thus we were combining two kinetic energies in laboratory space, each of magnitude K_1 = 6.08881 (ϕ_1 = 2.64665). From this, ϕ = 5.29330, which corresponds to K = 98.5025.

15.26 *Energy and mass*

The nonrelativistic expression for kinetic energy $T = \frac{1}{2}mu^2$ has just one term in it, a term which depends on the speed. The relativistic expression which approximates to the nonrelativistic expression at low speeds) can be written $T = mc^2 - m_0c^2$; that is, a speed-dependent term minus a constant term. The kinetic energy can be thought of as the excess over the energy over the constant term m_0c^2. The expression m_0c^2 is known as the *rest-mass energy*. The sum of the kinetic energy and the rest-mass energy is the "total energy", or just the "energy" E:

$$E = T + m_0c^2 = mc^2. \qquad 15.26.1$$

This means that, if the kinetic energy of a particle is zero, the total energy of the particle is not zero – it still has its rest-mass energy m_0c^2.

Of course, giving the name "rest-mass energy" to the constant term m_0c^2, and calling the speed-dependent term mc^2 the "total energy" and writing the famous equation $E = mc^2$, does not by itself immediately and directly tell us that "matter" can be converted to "energy" or the other way round. Whether such conversion can in fact take place is a matter for experiment and observation to determine. The equation by itself merely tells us how much mass is held by a given quantity of energy, or how much energy is held by a given quantity of mass. That entities that we traditionally think of as "matter" can be converted into entities that we traditionally think of as "energy" is well established with, for example, the "annihilation" of an electron and a positron ("matter" and "antimatter") to form photons ("energy") as is the inverse process of pair production (production of an electron-positron pair from a gamma ray in the presence of a third body).

It is unfortunate that the main (almost the only) example of application of the equation $E = mc^2$ persistently presented to the nonscientific public is the atom bomb, whose operation actually has nothing at all to do with the equation $E = mc^2$, nor, contrary to the popular mind, is any "matter" converted to energy.

I have heard it said that you can find out on the Web how to build an atom bomb, so here goes – here is how an atom bomb works. A uranium-235 nucleus is held together by strong attractive forces between the nucleons, which, at short femtometre ranges are much stronger than the Coulomb repulsive forces between the protons. When the nucleus absorbs an additional neutron, the resulting ^{236}U nucleus is unstable and breaks up into two intermediate-mass nuclei plus two or three neutrons. The two intermediate-mass nuclei are generally not of exactly equal mass; one is usually a bit less than half of the uranium nucleus and the other a bit more than half, but that's a detail. The potential energy required to bind the nucleons together in the uranium nucleus is rather greater than the binding energy of the two resulting intermediate-mass nuclei; the difference is of order 200 MeV, and that potential energy is converted into kinetic energy of the two resulting nuclei and, to a lesser extent, the two or three neutrons released. That is all. It is merely the familiar conversion of potential binding energy (admittedly a great deal of energy) into kinetic energy. No matter, no protons, no neutrons, are "destroyed" or "converted into energy", and $E = mc^2$ simply doesn't enter into it anywhere! The rest-mass energy of a proton or a neutron is about 1 GeV, and that much energy would be released if a proton were miraculously and for no cause converted into energy. Let us hope that no one invents a bomb that will do that – though we may rest assured that that is rather unlikely.

Where the equation $E = mc^2$ does come in is in the familiar observation that the mass of any nucleus other than hydrogen is a little less than the sum of the masses of the constituent nucleons. It is for that reason that nuclear masses, even for pure isotopes, are not integral. The mass of a nucleus is equal to the sum of the masses of the constituent nuclei plus the mass of the binding energy, the latter being a negative quantity since the inter-nucleon forces are attractive forces. The equation $E = mc^2$ tells us that energy (such as, for example, the binding energy between nucleons) has mass.

15.27 *Energy and Momentum*

A moving particle has energy arising from its momentum and also from its rest mass, and we need to find an expression relating energy to rest mass and momentum. It is fairly easy and it goes like this:

$$\begin{aligned} E^2 &= m^2c^4 = c^2(m^2c^2 - m^2u^2 + m^2u^2) = c^2[m^2(c^2 - u^2) + p^2] \\ &= c^2\left(\frac{m_0^2(c^2-u^2)}{1 - u^2/c^2} + p^2\right) = c^2(m_0^2c^2 + p^2). \end{aligned}$$

Thus we obtain for the energy in terms of rest mass and momentum

$$E^2 = (m_0c^2)^2 + (pc)^2. \qquad 15.27.1$$

If the speed (and hence momentum) is zero, the energy is merely m_0c^2. If the rest mass is zero (as, for example, a photon) and the energy is not zero, then $E = pc = muc$. But also $E = mc^2$, so that, if the rest mass of a particle is zero and the energy is not, the particle must be moving at the speed of light. This could be regarded as the reason why photons, which have zero rest mass, travel at the speed of photons. If neutrinos have zero rest mass, they, too, will travel at the speed of light; if they are not massless, they won't.

In addition to equation 15.27.1, which relates the energy to the magnitude of the momentum, it will be of interest to see how the *components* of momentum transform between reference frames. As usual, we are considering frame Σ' to be moving with respect to Σ at a speed υ with respect to Σ. There is no difficulty with the *y*- and *z*- components. We have merely $p'_{y'} = p_y$ and $p'_{z'} = p_z$. However:

$$p_x = mu_x = \frac{m_0 u_x}{\left(1 - u_x^2/c^2\right)^{1/2}} \quad \text{and} \quad p'_{x'} = m'u'_{x'} = \frac{m_0 u'_{x'}}{\left(1 - u_{x'}^2/c^2\right)^{1/2}}.$$

Also $\quad u'_{x'} = \dfrac{u_x - \upsilon}{\left(1 - u_x^2/c^2\right)}, \quad$ from which $\quad (1 - u'^2_{x'}/c^2)^{1/2} = \dfrac{(1 - u_x^2/c^2)^{1/2}(1 - \upsilon^2/c^2)^{1/2}}{1 - u_x\upsilon/c^2}.$

After a little algebra, we obtain

$$p_x = \frac{m_0(u_x - \upsilon)}{(1 - u^2/c^2)^{1/2}(1 - \upsilon^2/c^2)^{1/2}}.$$

And this is

$$p'_{x'} = \frac{p_x - \upsilon E/c^2}{(1 - \upsilon^2/c^2)^{1/2}} = \gamma(p_x - \upsilon E/c^2). \qquad 15.27.2$$

The inverse is found in the usual way:

$$p_x = \gamma(p'_{x'} + \upsilon E'/c^2). \qquad 15.27.3$$

If we eliminate $p'_{x'}$ from equations 15.27.2 and 15.27.3, we'll find *E*' in terms of *E* and p_x:

$$E' = \gamma(E - \upsilon p_x). \qquad 15.27.4$$

Thus the transformations between energy and the three spatial components of momentum is similar to the transformation between time and the three space coordinates, and are described by a similar 4-vector:

$$\begin{pmatrix} p'_{x'} \\ p'_{y'} \\ p'_{z'} \\ iE'/c \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \\ iE/c \end{pmatrix}. \qquad 15.27.5$$

The reader should multiply this out to verify that it does reproduce equations 15.27.3 and 15.27.4.

15.28 *Units*

It is customary in the field of particle physics to express energy (whether total, kinetic or rest-mass energy) in electron volts (eV) or in keV, MeV, GeV or TeV (10^3, 10^6, 10^9, or 10^{12} eV respectively). A electron volt is the kinetic energy gained by an electron if it is accelerated through an electrical potential of 1 volt; alternatively it is the work required to move an electron through one volt. Either way, since the charge on an electron is 1.602×10^{-19}) C, 1eV = 1.602×10^{-19} J.

The use of such a unit may understandably dismay those who would insist always on expressing any physical quantity in SI units, and I am much in sympathy with this view. Yet, to those who deal daily with particles whose charge is equal to or is a small multiple or rational fraction of the electronic charge, the eV has its attractions. Thus if you accelerate a particle through so many volts, you don't have to remember the exact value of the electronic charge or carry out a long multiplication every time you do so. One might also think of a hypothetical question such as: An electron is accelerated through 3426.7189628471 volts. What is its gain in kinetic energy? You *cannot answer this in joules* unless you know the value of the electronic charge to a comparable precision; but of course you do know the answer in eV.

One situation that does require care is this. An α-particle is accelerated through 1000 V. What is the gain in kinetic energy? Because the charge on an α-particle is twice that of an electron, the answer is 2000 eV.

Very often you know the energy of a particle (because you have accelerated it through so many volts) and you want to know its momentum; or you know its momentum (because you have measured the curvature of its path in a magnetic field) and you want to know its energy. Thus you will frequent occasion to make use of equation 15.27.1:

$$E^2 = (m_0c^2)^2 + (pc)^2 .$$

You have to be careful to remember how many *c*s there are, and what is the exact value of *c*. Particle physicists prefer to make life easier for themselves (not necessarily for the rest of us!) by preferring not to state what the momentum of a particle is, or its rest mass, but rather to give the values of pc or of m_0c^2 – and to express E, pc and m_0c^2 all in eV (or keV, MeV or GeV). Thus one may hear that

$$pc = 6.2 \text{ GeV}$$
$$m_0c^2 = 0.938 \text{ GeV}.$$

More often this is expressed, somewhat idiosyncratically and in somewhat doubtful use of English, as

$$p = 6.2 \text{ GeV}/c$$
$$m_0 = 0.938 \text{ GeV}/c^2$$

or in informal casual conversation (one hopes not for publication) merely as

$$p = 6.2 \text{ GeV}$$
$$m_0 = 0.938 \text{ GeV}.$$

While this may puzzle some and raise the ire of others, it is not entirely without merit, because, provided one uses these units, the relation between energy, momentum and rest mass is then simply

$$E^2 = m_0^2 + p^2.$$

The practice is not confined to energy, momentum and rest mass. For example, the SI unit of magnetic dipole moment is N m T^{-1} (newton metre per tesla). Now N m (unit of torque) is not quite the same as a joule (unit of energy), although dimensionally similar. Yet it is common practice to express the magnetic moments of subatomic particles in eV T^{-1}. Thus the Bohr magneton is a unit of magnetic dipole moment equal to 9.27×10^{-24} N m T^{-1}, and this may be expressed as 5.77×10^{-5} eV T^{-1}.

One small detail to be on guard for is this. One may hear talk of "a 500 MeV proton". Does this mean that the *kinetic* energy is 500 MeV or that its *total* energy is 500 MeV? In this case the answer is fairly clear (although it would have been completely clear if the speaker had been explicit). The rest-mass energy of a proton is 938 MeV, so he must have been referring to the kinetic energy. If, however, he had said "a 3 GeV proton", there would be no way of deducing whether he was referring to the kinetic or the total energy. And if he had said "a 3 GeV particle", there would be no way of telling whether he was referring to its total energy, its kinetic energy or its rest-mass energy. It is incumbent on all of us – or at least those of us who wish to be understood by others – always to make ourselves explicitly clear and not to suppose that others will correctly guess what we mean.

15.29 *Force*

Force is defined as rate of change of momentum, and we wish to find the transformation between forces referred to frames in uniform relative motion such that this relation holds on all such frames.

Suppose that, in Σ', a mass has instantaneous mass *m*' and velocity whose instantaneous components are $u'_{x'}$ *and* $u'_{y'}$. If a force acts on it, then the velocity *and hence also the mass* are functions of time. The *x*-component of the force is given by

$$F'_{x'} = \frac{d}{dt'}\left(m'u'_{x'}\right). \qquad 15.29.1$$

We want to express everything on the right hand side in terms of unprimed quantities. Thus from equation 15.21.8 and the inverse of equation 15.16.2, we obtain

$$m'u'_{x'} = m\gamma\left(u_x - \upsilon\right). \qquad 15.29.2$$

Also

$$\frac{d}{dt'} \equiv \frac{dt}{dt'}\frac{d}{dt} \qquad 15.29.3$$

Let us first evaluate $\frac{d}{dt}(m\gamma u_x - m\gamma\upsilon)$. In this expression, υ and γ are independent of time (the frame Σ' is moving at constant velocity relative to Σ), and $\frac{d}{dt}$ of mu_x is the *x*-component of the force in Σ, that is F_x. Thus

$$\frac{d}{dt}(m\gamma u_x - m\gamma\upsilon) = \gamma\left(F_x - \upsilon\frac{dm}{dt}\right). \qquad 15.29.4$$

Now we need to evaluate $\frac{dt}{dt'}$ *in terms of unprimed quantities.* If we start with

$$dt = \left(\frac{\partial t}{\partial x'}\right)_{t'} dx' + \left(\frac{\partial t}{\partial t'}\right)_{x'} dt' \qquad 15.29.5$$

we'll just get more primed quantities. What we'll do instead is to start with

$$dt' = \left(\frac{\partial t'}{\partial x}\right)_{t} dx + \left(\frac{\partial t'}{\partial t}\right)_{x} dt \qquad 15.29.6$$

and we'll evaluate $\frac{dt'}{dt}$, which, being a total derivative, is the reciprocal of $\frac{dt}{dt'}$. The partial derivatives are given by equations 15.15.3k and l, while $\frac{dx}{dt} = u_x$. Hence we obtain

$$\frac{dt}{dt'} = \frac{1}{\gamma\left(1 - \frac{u_x \upsilon}{c^2}\right)} . \qquad 15.29.7$$

Thus we arrive at

$$F'_{x'} = \frac{F_x - \upsilon(dm/dt)}{1 - u_x \upsilon / c^2} . \qquad 15.29.8$$

The mass is not constant (i.e. *dm*/*dt* is not zero) because there is a force acting on the body, and we have to relate the term *dm*/*dt* to the force. At some instant when the force and velocity (in Σ) are **F** and **u**, the rate at which **F** is doing work on the body is $\mathbf{F} \bullet \mathbf{u} = F_x u_x + F_y u_y + F_z u_z$ and this is equal to the rate of increase of energy of the body, which is $\dot{m}c^2$. (In section 15.24, in deriving the expression for kinetic energy, I wrote that the rate of doing work was equal to the rate of increase of *kinetic* energy. Now I have just written that it is equal to the rate of increase of (total) energy. Which is right?)

$$\therefore \qquad \frac{dm}{dt} = \frac{1}{c^2}(F_x u_x + F_y u_y + F_z u_z). \qquad 15.29.9$$

Substitute this into equation 15.29.8 and, after a very little more algebra, we finally obtain the transformation for $F'_{x'}$:

$$F'_{x'} = F_x - \frac{\upsilon}{c^2 - u_x \upsilon}(u_y F_y + u_z F_z). \qquad 15.29.10$$

The y'- and z'- components are a little easier, and I leave it as an exercise to show that

$$F'_{y'} = \frac{\sqrt{1 - \upsilon^2 / c^2}}{1 - u_x \upsilon / c} F_y \qquad 15.29.11$$

$$F'_{z'} = \frac{\sqrt{1 - \upsilon^2 / c^2}}{1 - u_x \upsilon / c} F_z . \qquad 15.29.12$$

As usual, the inverse transformations are found by interchanging the primed and unprimed quantities and changing the sign of υ.

The force on a particle and its resultant acceleration are not in general in the same direction, because the mass is not constant. (Newton's second law is not $\mathbf{F} = m\mathbf{a}$; it is $\mathbf{F} = \dot{\mathbf{p}}$.) Thus

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{u}) = m\mathbf{a} + \dot{m}\mathbf{u}. \qquad 15.29.13$$

Here $$m = \frac{m_0}{(1 - u^2/c^2)^{1/2}} \qquad 15.29.14$$

and so $$\dot{m} = \frac{m_0 u a}{c^2(1 - u^2/c^2)^{3/2}}.$$

15.29.15

Thus $$\mathbf{F} = \frac{m_0}{(1 - u^2/c^2)^{1/2}}\left(\mathbf{a} + \frac{u a}{c^2 - u^2}\mathbf{u}\right). \qquad 15.29.16$$

15.30 *Electromagnetism*

These notes are intended to cover only mechanics, and therefore I resist the temptation to cover here special relativity and electromagnetism. I point out only that in many ways this misses many of the most exciting parts of special relativity, and indeed it was some puzzles with electromagnetism that led Einstein to formulate the theory of special relativity. One proceeds as we have done with mechanical quantities; that is, we have to define carefully what is meant by each quantity and how in principle it is possible to measure it, and then see how it transforms between frames in such a manner that the laws of physics – in particular Maxwell's equations - are the same in each. One such transformation that is found, for example, is $\mathbf{E}' = \gamma(\mathbf{E} + \mathbf{u}\times\mathbf{B})$, so that what appears in one frame as an electric field appears in another at least in part as a magnetic field. The Coulomb force transforms to a Lorentz force; Coulomb's law transforms to Ampères law.

Although I do no more than mention this topic here, I owe it to the reader to say just a little bit more about the speedometer that I designed in section 15.4. It is indeed true that, as the train moves forward, the net repulsive force between the two rods does diminish, although not quite as I have indicated, for one has to make the correct transformations between frames for force, current, electric field, magnetic field, and so on. But it turns out that the weights of the rods – i.e. the downward forces on them – also diminish in exactly the same ratio, and the angle between the strings remains stubbornly the same. Our trip to the patent office will be in vain. The speedometer will not work, and it remains impossible to determine the absolute motion of the train.