CHAPTER 14
HAMILTONIAN MECHANICS

14.1 *Introduction*

The hamiltonian equations of motion are of deep theoretical interest. Having established that, I am bound to say that I have not been able to think of a problem in classical mechanics that I can solve more easily by hamiltonian methods than by newtonian or lagrangian methods. That is not to say that real problems cannot be solved by hamiltonian methods. What I have been looking for is a problem which I can solve easily by hamiltonian methods but which is more difficult to solve by other methods. So far, I have not found one. Having said that, doubt not that hamiltonian mechanics is of deep theoretical significance.

Having expressed that mild degree of cynicism, let it be admitted that Hamilton theory – or more particularly its extension the Hamilton-Jacobi equations – does have applications in celestial mechanics, and of course hamiltonian operators play a major part in quantum mechanics, although it is doubtful whether Sir William would have recognized his authorship in that connection.

14.2 *A Thermodynamics Analogy*

Readers may have noticed from time to time – particularly in Chapter 9 – that I have perceived some connection between parts of classical mechanics and thermodynamics. I perceive such an analogy in developing hamiltonian dynamics. Those who are familiar with thermodynamics may also recognize the analogy. Those who are not can skip this section without seriously prejudicing their understanding of subsequent sections.

Please do not misunderstand: The hamiltonian in mechanics is not at all the same thing as enthalpy in thermodynamics, even though we use the same symbol, H. Yet there are similarities in the way we can introduce these concepts.

In thermodynamics we can describe the state of the system by its internal energy, defined in such a way that when heat is supplied **to** a system and the system does external work, the **increase** in internal energy of the system is equal to the heat supplied **to** the system minus the work done **by** the system:

$$dU = T\,dS - P\,dV . \qquad 14.2.1$$

From this point of view we are describing the state of the system by specifying its internal energy as a function of the entropy and the volume:

$$U = U(S, V) \qquad 14.2.2$$

so that
$$dU = \left(\frac{\partial U}{\partial S}\right)_V dS + \left(\frac{\partial U}{\partial V}\right)_S dV, \qquad 14.2.3$$

from which we see that
$$T = \left(\frac{\partial U}{\partial S}\right)_V \quad \text{and} \quad -P = \left(\frac{\partial U}{\partial V}\right)_S . \qquad 14.2.4,5$$

However, it is sometimes convenient to change the basis of the description of the state of a system from S and V to S and P by defining a quantity called the enthalpy H defined by

$$H = U + PV. \qquad 14.2.6$$

In that case, if the state of the system changes, then

$$dH = dU + P\,dV + V\,dP \qquad 14.2.7$$

$$= T\,dS - P\,dV + P\,dV + V\,dP. \qquad 14.2.8$$

I.e.
$$dH = T\,dS + V\,dP. \qquad 14.2.9$$

Thus we see that, if heat is added to a system held at constant *volume*, the increase in the *internal energy* is equal to the heat added; whereas if heat is added to a system held at constant *pressure*, the increase in the *enthalpy* is equal to the heat added.

From this point of view we are describing the state of the system by specifying its enthalpy as a function of the entropy and the pressure:

$$H = H(S, P) \qquad 14.2.10$$

so that
$$dH = \left(\frac{\partial H}{\partial S}\right)_P dS + \left(\frac{\partial H}{\partial P}\right)_S dP, \qquad 14.2.11$$

from which we see that

$$T = \left(\frac{\partial H}{\partial S}\right)_P \quad \text{and} \quad V = \left(\frac{\partial H}{\partial P}\right)_S . \qquad 14.2.12$$

None of this has anything to do with hamiltonian dynamics, so let's move on.

14.3 *Hamilton's Equations of Motion*

In classical mechanics we can describe the state of a system by specifying its lagrangian as a function of the coordinates and their time rates of change:

$$L = L(q_i, \dot{q}) \qquad 14.3.2$$

(I am deliberately numbering this equation 14.3.2, to maintain an analogy between this section and section 14.2.)

If the coordinates and the velocities increase, the corresponding increment in the lagrangian is

$$dL = \sum_i \frac{\partial L}{\partial q_i} dq_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i . \qquad 14.3.3$$

Definition: The *generalized momentum* p_i associated with the generalized coordinate q_i is defined as

$$p_i = \frac{\partial L}{\partial \dot{q}_i} . \qquad 14.3.4$$

[You have seen this before, in Section 13.4 of Chapter 13. Remember "ignorable coordinate"?]

It follows from the lagrangian equation of motion $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$ (equation 13.4.14) that

$$\dot{p}_i = \frac{\partial L}{\partial q_i} . \qquad 14.3.5$$

Thus $$dL = \sum_i \dot{p}_i \, dq_i + \sum_i p_i \, d\dot{q}_i . \qquad 14.3.1$$

(I am deliberately numbering this equation 14.3.1, to maintain an analogy between this section and section 14.2.)

However, it is sometimes convenient to change the basis of the description of the state of a system from q_i and $\dot{q}_i$ to q_i and $\dot{p}_i$ by defining a quantity called the hamiltonian H defined by

$$H = \sum_i p_i \dot{q}_i - L. \qquad \underline{\textit{Definition}} \qquad 14.3.6$$

In that case, if the state of the system changes, then

$$dH = \sum_i p_i d\dot{q}_i + \sum_i \dot{q}_i dp_i - dL \qquad 14.3.7$$

$$= \sum_i p_i d\dot{q}_i \ + \ \sum_i \dot{q}_i dp_i \ - \ \sum_i \dot{p}_i dq_i \ - \ \sum_i p_i d\dot{q}_i \,. \qquad 14.3.8$$

I.e.
$$dH \ = \ \sum_i \dot{q}_i dp_i \ - \ \sum_i \dot{p}_i dq_i \,. \qquad 14.2.9$$

We are regarding the hamiltonian as a function of the generalized coordinates and generalized momenta:

$$H \ = \ H(q_i \, , p_i), \qquad 14.3.10$$

so that
$$dH \ = \ \sum_i \frac{\partial H}{\partial q_i} dq_i + \sum_i \frac{\partial H}{\partial p_i} dp_i \,, \qquad 14.3.11$$

from which we see that

$$-\dot{p}_i \ = \ \frac{\partial H}{\partial q_i} \quad \text{and} \quad \dot{q}_i \ = \ \frac{\partial H}{\partial p_i} \qquad 14.3.12,13$$

In summary, then, equations 14.3.4,5,12 and 13:

$$\boxed{\begin{array}{ll} p_i \ = \ \dfrac{\partial L}{\partial \dot{q}_i} & \dot{p}_i \ = \ \dfrac{\partial L}{\partial q_i} \\[2em] -\dot{p}_i \ = \ \dfrac{\partial H}{\partial q_i} & \dot{q}_i \ = \ \dfrac{\partial H}{\partial p_i} \end{array}}$$

which I personally find impossible to commit accurately to memory (although note that there is one dot in each equation) except when using them frequently, may be regarded as Hamilton's equations of motion. I'll refer to these equations as **A**, **B**, **C** and **D**.

Note that, in the second equation, if the lagrangian is independent of the coordinate q_i, the coordinate q_i is referred to as an "ignorable coordinate". I suppose it is called "ignorable" because you can ignore it when calculating the lagrangian, but in fact a so-called "ignorable" coordinate is usually a very interesting coordinate indeed, because it means (look at the second equation) that the corresponding generalized momentum is conserved.

Now the kinetic energy of a system is given by $T = \frac{1}{2}\sum_i p_i \dot{q}_i$ (for example, $\frac{1}{2}mvv$), and the hamiltonian (equation 14.3.6) is defined as $H = \sum_i p_i \dot{q}_i - L$. For a *conservative system*, $L = T - V$, and hence, for a conservative system, $H = T + V$. If you are asked in an examination to explain what is meant by the hamiltonian, by all means say it is $T + V$. That's fine for a conservative system, and you'll probably get half marks. That's 50% - a D grade, and you've passed. If you want an A+, however, I recommend equation 14.3.6.

14.4 *Examples*

I'll do two examples by hamiltonian methods – the simple harmonic oscillator and the soap slithering in a conical basin. Both are conservative systems, and we can write the hamiltonian as $T + V$, but we need to remember that we are regarding the hamiltonian as a function of the generalized coordinates and *momenta*. Thus we shall generally write translational kinetic energy as $p^2/(2m)$ rather than as $\frac{1}{2}mv^2$, and rotational kinetic energy as $L^2/(2I)$ rather than as $\frac{1}{2}I\omega^2$.

Simple harmonic oscillator

The potential energy is $\frac{1}{2}kx^2$, so the hamiltonian is

$$H = \frac{p^2}{2m} + \tfrac{1}{2}kx^2.$$

From equation **D**, we find that $\dot{x} = p/m$, from which, by differentiation with respect to the time, $\dot{p} = m\ddot{x}$. And from equation **C**, we find that $\dot{p} = -kx$. Hence we obtain the equation of motion $m\ddot{x} = -kx$.

Conical basin

We refer to section 13.6 of chapter 13.

$$T = \tfrac{1}{2}m(\dot{r}^2 + r^2\sin^2\alpha\,\dot{\phi}^2),$$
$$V = mgr\cos\alpha,$$
$$L = \tfrac{1}{2}m(\dot{r}^2 + r^2\sin^2\alpha\,\dot{\phi}^2) - mgr\cos\alpha,$$
$$H = \tfrac{1}{2}m(\dot{r}^2 + r^2\sin^2\alpha\,\dot{\phi}^2) + mgr\cos\alpha.$$

But, in the hamiltonian formulation, we have to write the hamiltonian in terms of the generalized momenta, and we need to know what they are. We can get them from the lagrangian and equation **A** applied to each coordinate in turn. Thus

$$p_r = \frac{\partial L}{\partial \dot r} = m\dot r \qquad \text{and} \qquad p_\phi = \frac{\partial L}{\partial \dot\phi} = mr^2 \sin^2\alpha\,\dot\phi. \qquad 14.4.1,2$$

Thus the hamiltonian is

$$H = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2\sin^2\alpha} + mgr\cos\alpha. \qquad 14.4.3$$

Now we can obtain the equations of motion by applying equation **D** in turn to r and ϕ and then equation **C** in turn to r and ϕ:

$$\dot r = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}, \qquad 14.4.4$$

$$\dot\phi = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2\sin^2\alpha}, \qquad 14.4.5$$

$$\dot p_r = -\frac{\partial H}{\partial r} = \frac{p_\phi^2}{mr^3\sin^2\alpha} - mg\cos\alpha, \qquad 14.4.6$$

$$\dot p_\phi = \frac{\partial H}{\partial \phi} = 0. \qquad 14.4.7$$

Equations 14.4.2 and 7 tell us that $mr^2\sin^2\alpha\,\dot\phi$ is constant and therefore that

$$r^2\dot\phi \text{ is constant, } = h\text{, say.} \qquad 14.4.8$$

This is one of the equations that we arrived at from the lagrangian formulation, and it expresses constancy of angular momentum.

By differentiation of equation 14.4.1 with respect to time, we see that the left hand side of equation 14.4.6 is $m\ddot r$. On the right hand side of equation 14.4.6, we have p_ϕ, which is constant and equal to $mh\sin^2\alpha$. Equation 14.4.6 therefore becomes

$$\ddot r = \frac{h^2\sin^2\alpha}{r^3} - g\cos\alpha, \qquad 14.4.9$$

which we also derived from the lagrangian formulation.

14.5 *Poisson Brackets*

Let f and g be functions of the generalized coordinates and momenta. Think first of all of one coordinate, say q_i, and its conjugate momentum p_i (defined, you may remember, as $\partial L/\partial \dot{q}_i$). I now ask the question: Is $\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}$ the same thing as $\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$?

After thinking about it you will probably say something like: Well, I dare say that you *might be able to find* two functions such that that is so, but I don't see why it should be so for *any* two arbitrary functions. If that is what you thought, you thought right. Pairs of functions such that these two expressions are equal are of special significance. And pairs of functions such that these two expressions are *not* equal are also of special significance

The *Poisson bracket* of two functions of the coordinates and momenta is defined as

$$[f, g,] \;=\; \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right). \qquad 14.5.1$$

(Poisson brackets are sometimes written as *braces* - i.e. $\{\}$. I'm not sure whether braces $\{\}$ or brackets [] are the commoner. I have chosen brackets here, so that I don't have to call them Poisson braces.)

Poisson brackets have important applications in celestial mechanics and in quantum mechanics. In celestial mechanics, they are used in the developments of *Lagrange's planetary equations*, which are used to calculate the perturbations of the elements of the planetary orbits under small deviations from ideal two-body point-source orbits. See, for example, Chapter 14 of the Celestial Mechanics set of these notes. Readers who have had an introductory course in quantum mechanics may have come across the *commutator* of two operators, and will (or should!) understand the significance of two operators that commute. (It means that a function can be found that is simultaneously an eigenfunction of both operators.) You may not have thought of the commutator as being a *Poisson bracket*, but you soon will.

Let's suppose (because it doesn't make any essential difference) that there is just a single generalized coordinate and its conjugate generalized momentum, so that the Poisson bracket is just

$$[f, g] = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}. \qquad 14.5.2$$

Now let's suppose that f is just q, the coordinate, and that g is the Hamiltonian, H, which is defined, you will recall, as $p\dot{q} - L$, and is a function of the coordinate and the momentum. What, then is the Poisson bracket $[q, H]$?

Answer: $$[q,H] = \frac{\partial q}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial q}{\partial p}\frac{\partial H}{\partial q}. \qquad 14.5.3$$

The coordinate and the momentum are independent variables, so that $\partial q / \partial p$ is zero, so the second term on the right hand side of equation 14.5.3 is zero. In the first term on the right hand side, $\partial q / \partial q$ is of course 1, and $\partial H / \partial p$, by Hamilton's equations of motion, is $\dot{q}$. Thus, the answer is

$$[q,H] = \dot{q}. \qquad 14.5.4$$

In a similar vein, you will find (DO IT!!) that

$$[p,H] = \dot{p}. \qquad 14.5.5$$

Thus neither the generalized coordinate nor the generalized momentum commutes with the Hamiltonian.

Now go a little further, and suppose that there are more than one coordinate and more than one momentum. Two will do, so that

$$[f,g] = \frac{\partial f}{\partial q_1}\frac{\partial g}{\partial p_1} - \frac{\partial f}{\partial p_1}\frac{\partial g}{\partial q_1} + \frac{\partial f}{\partial q_2}\frac{\partial g}{\partial p_2} - \frac{\partial f}{\partial p_2}\frac{\partial g}{\partial q_2}. \qquad 14.5.6$$

Can you show that:

$$[p_1, p_2] = [q_1, q_2] = [p_1, q_2] = [q_1, p_2] = 0; \qquad [q_1, p_1] = 1.? \qquad 14.5.7$$

I shan't go any further than that here, because it would take us too far into quantum mechanics. However, those readers who have done some introductory quantum mechanics may recall that there are various pairs of operators that do or do not commute, and may now begin to appreciate the relation between the Poisson brackets of certain pairs of observable quantities and the commutator of the operators representing these quantities. For example, consider the last of these. It shows that a coordinate such as x does not commute with its corresponding momentum p_x. There is nothing more certain that this. So certain is it that it ought to be called Heisenberg's Certainty Principle. But for some reason people often seem to present quantum mechanics as something uncertain or mysterious, whereas in reality there is nothing uncertain or mysterious about it at all.