CHAPTER 10
ROCKET MOTION

1. *Introduction*

If you are asked to state Newton's Second Law of Motion, I hope you will not reply: "Force equals mass times acceleration" − because that is not Newton's Second Law of Motion. Newton's Second Law of Motion is:

The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

In short: Force equals Rate of Change of Momentum, or, in symbols, $F = \dot{p}$. On differentiating the right hand side, we obtain $F = m\dot{v} + \dot{m}v$. In other words, if the mass is constant then indeed force equals mass times acceleration − but *only* if the mass is constant. In a rocket, a very appreciable fraction of the mass of the rocket is fuel, which is burned and ejected at a very high rate, so that the mass of the rocket is rapidly diminishing during the motion. It is one of the great problems of rocket design that such a high proportion of the initial mass must be fuel. For this reason, other possible methods of driving spacecraft are being investigated by many groups. For example, in the ion propulsion system of the Deep Space One spacecraft, electrically accelerated ions are ejected at high speed from the spacecraft. The force produced and the acceleration are minute, but, because it can be kept up for a very long time, very high speeds can eventually be reached. "Solar sail" systems similarly rely on the very tiny force that can be exerted by the solar wind, but this tiny force can be exerted during most of the lifetime of a spacecraft's flight, and hence again high speeds can be reached.

This chapter, however, concerns just conventional rocket motion. In the next section I consider the motion of a rocket in space subject only to the one force from the high-speed ejection of burned fuel in the absence of any other forces. At a later date, if I can find the time and energy, I may add further sections on rocket motion against gravity, which might be uniform or might fall off with distance from Earth, and we might include air resistance or not. But to begin with, we deal solely with a rocket isolated in space and subject to no additional forces.

2. *An Integral*

So that we don't get bogged down later with an integral that is going to crop up, see if you can do the following integration:

$$\int \ln(a - bt)dt.$$

You should get $-t - \frac{1}{b}(a - bt)\ln(a - bt) +$ constant.

3. *The Rocket Equation.*

Initially at time $t = 0$, the mass of the rocket, including fuel, is m_0.

We suppose that the rocket is burning fuel at a rate of b kg s^{-1} so that, at time t, the mass of the rocket-plus-remaining-fuel is $m = m_0 - bt$. The rate of increase of mass with time is $\frac{dm}{dt} = -b$ and is supposed constant with time. (The rate of "increase" is, of course, negative.)

We suppose that the speed of the ejected fuel, relative to the rocket, is V. The thrust of the ejected fuel on the rocket is therefore Vb, or $-V\frac{dm}{dt}\cdot$ This is equal to the instantaneous mass times acceleration of the rocket:

$$Vb = m\frac{dv}{dt} = (m_0 - bt)\frac{dv}{dt}\cdot \qquad 10.3.1$$

Thus

$$\int_0^v dv = Vb\int_0^t \frac{dt}{m_0 - bt}\cdot \qquad 10.3.2$$

(Don't be tempted to write the right hand side as $-Vb\int_0^t \frac{dt}{bt - m_0}\cdot$ You are anticipating a logarithm, so keep the denominator positive. We have met this before in Chapter 6.) On integration, we obtain

$$v = V\ln\frac{m_0}{m_0 - bt}\cdot \qquad 10.3.3$$

The acceleration is

$$\frac{dv}{dt} = \frac{Vb}{m_0 - bt}\cdot \qquad 10.3.4$$

At $t = 0$, the speed is zero and the acceleration is Vb/m_0.

At time $t = m_0/b$, the remaining mass is zero and the speed and acceleration are both infinite. However, this is so only if the initial mass is 100% fuel and nothing else. This is not realistic. If the fraction of the total mass was initially f, the fuel will be completely expended after a time fm_0/b at which time the speed will be $-V\ln(1-f)$ (which is, of course, positive), and the speed will remain constant thereafter. For example, if $f = 99\%$, the final speed will be $4.6V$.

Equations 10.3.3 and 10.3.4 are shown in figures X.1 and X.2. In figure X.1, the speed of the rocket is plotted in units of V, the ejection speed of the burnt fuel. The time is plotted in units of m_0/b. The fuel initially comprised 90% of the rocket, so that the rocket runs out of fuel in time 0.9 m_0/b, at which time its speed is $2.3V$. In figure X.2, the acceleration is plotted in units of the initial acceleration, which is Vb/m_0. When the fuel is exhausted, the acceleration is ten times this.

FIGURE X.1

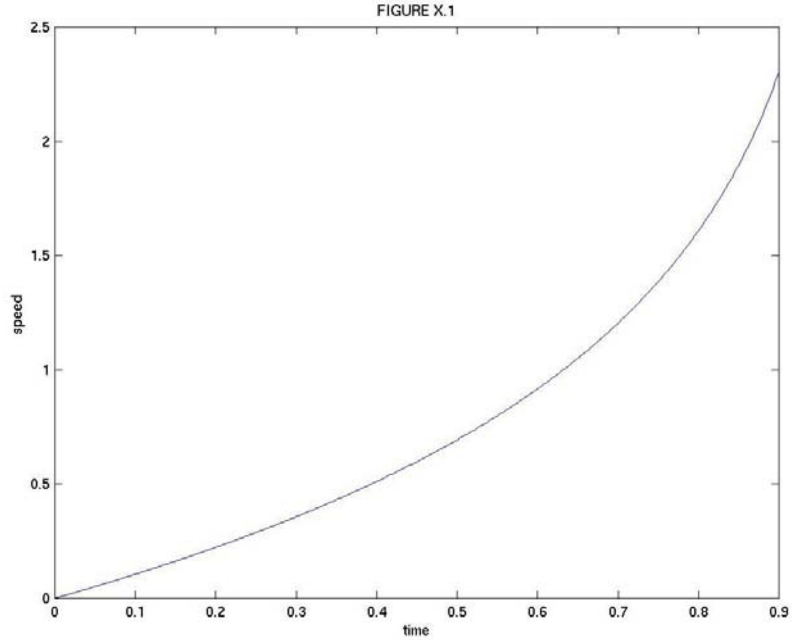
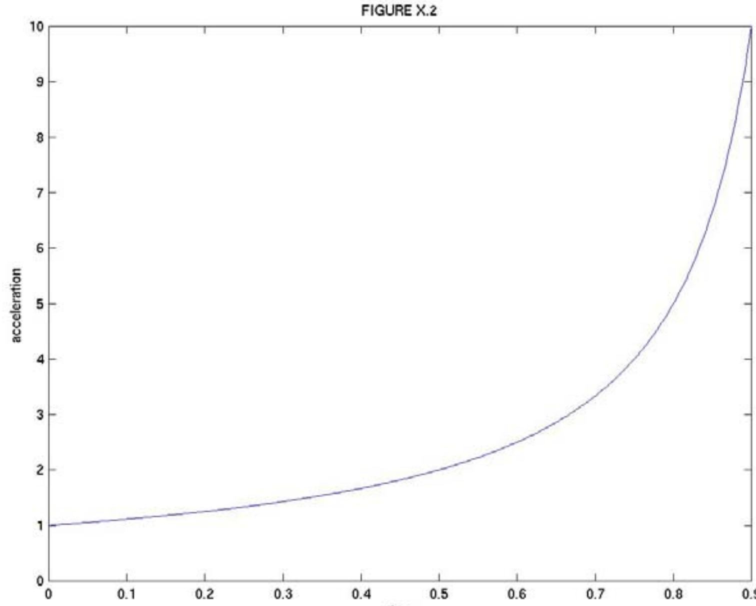


FIGURE X.2



In equation 10.3.3, υ is of course dx/dt, so the equation can be integrated to obtain the distance:time relation:

$$x = V\left[t + \left(\frac{m_0}{b} - t\right)\ln\left(1 - \frac{bt}{m_0}\right)\right]. \qquad 10.3.5$$

Elimination of t between equations 10.3.3 and 10.3.5 gives the relation between speed and distance:

$$x = \frac{Vm_0}{b}\left[1 - \left(1 + \frac{\upsilon}{V}\right)e^{-\upsilon/V}\right]. \qquad 10.3.6$$

If f is the fraction of the initial mass that is fuel, the fuel supply will be exhausted after a time fm_0/b, at which time its speed will be $-V\ln(1-f)$, (this is positive, because $1-f$ is less than 1), its acceleration will be $1/(1-f)$ and it will have travelled a distance $\frac{Vm_0}{b}[f + (1-f)\ln(1-f)]$. If the entire initial mass is fuel, so that $f = 1$, the fuel will burn for a time m_{0i}/b, at which time its speed and acceleration will be infinite, it will have travelled a finite distance Vm_0/b and the mass will have been reduced to zero, This remarkable result is not very believable, for two reasons. In the first place it is not very realistic. More importantly, when the speed becomes comparable to the speed of light, the equations which we have developed for nonrelativistic speeds are no longer approximately valid, and the correct relativistic equations must be used. The speed cannot then reach the speed of light as long as the remaining mass is non-zero.

Equations 10.3.5 and 10.3.6 are illustrated in figures X.3 and X.4, in which f, the fraction of the initial mass that is fuel, is 0.9. The units for distance, time and speed in these graphs are, respectively, Vm_0/b, m_0/b and V.

FIGURE X.4

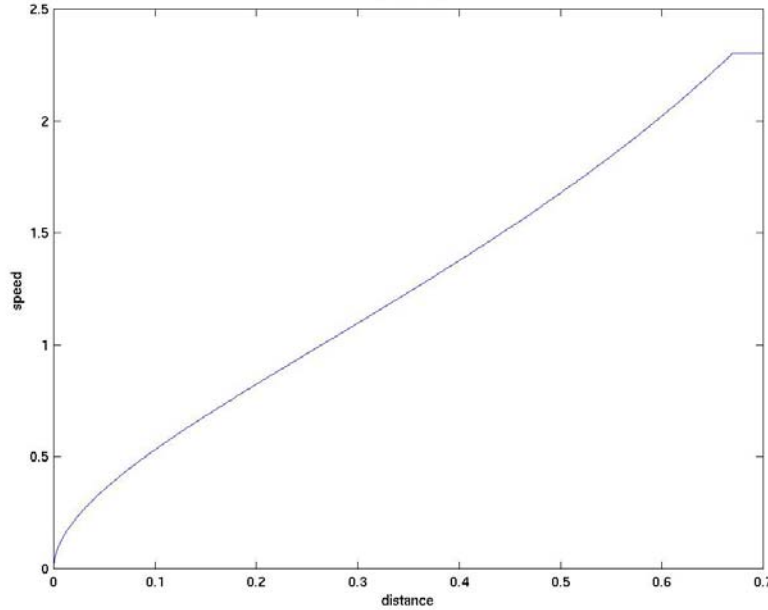
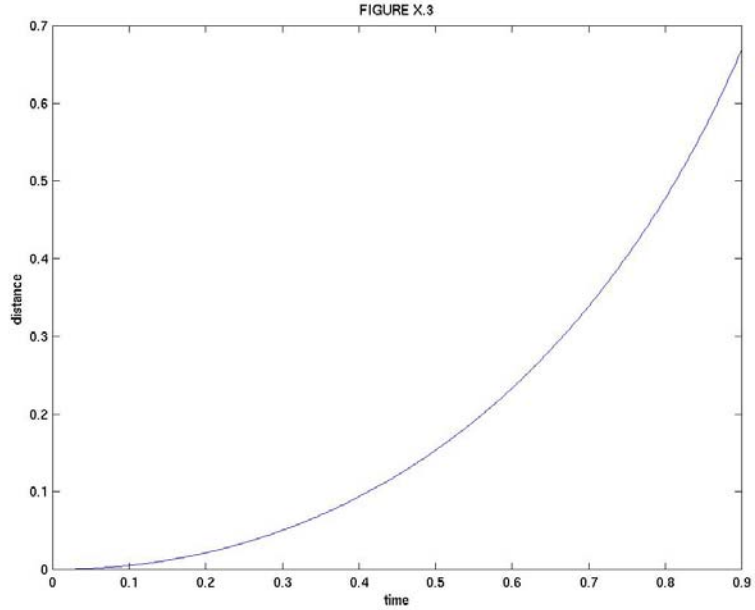


FIGURE X.3



4. *Problems*

(i) Derive the integral in section 2.
(ii) Integrate equation 10.3.2 to obtain equation 10.3.3
(iii) Integrate equation 10.3.3 to obtain equation 10.3.5
(iv) Obtain equation 10.3.6

In the following problems, (numbers v - viii) assume $V = 2$ km s^{-1}.
$m_0 = 2000$ kg
$b = 0.5$ kg s^{-1}
$f = 90\%$

(v) What is the maximum speed, and how long does it take to attain it?
(vi) How long does it take to reach a speed of 3 km s^{-1}?
(vii) How long does it take for the rocket to travel 600 km?
(viii) How fast is it moving when it has travelled 300 km?